Three-Dimensional Space and Vectors

Jeeja A. V.
Assistant Professor
Department of Mathematics
Govt. K.N.M Arts and Science College, Kanjiramkulam

E-Resource of Mathematics

FDP in MATHEMATICS under Choice Based Credit and Semester System, University of Kerala

According to the syllabus for 2018 Admission

Semester - III

MM 1341: Elementary Number Theory and Calculus – I - Module II

Contents

In this unit we will discuss rectangular coordinate systems in three dimensions, and we will study the analytic geometry of lines, planes, and other basic surfaces. The second theme of this unit is the study of vectors. These are the mathematical objects that physicists and engineers use to study forces, displacements, and velocities of objects moving on curved paths. More generally, vectors are used to represent all physical entities that involve both a magnitude and a direction for their complete description. We will introduce various algebraic operations on vectors, and we will apply these operations to problems involving force, work, and rotational tendencies in two and three dimensions. Finally, we will discuss cylindrical and spherical coordinate systems, which are appropriate in problems that involve various kinds of symmetries and also have specific applications in navigation and celestial mechanics.

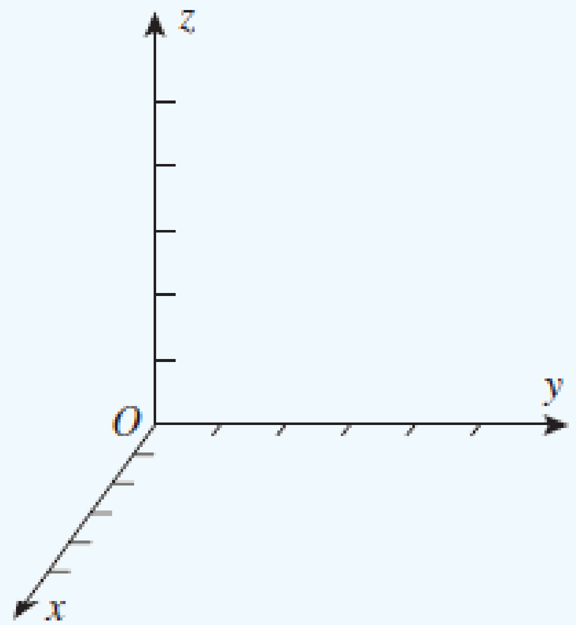
1 THREE-DIMENSIONAL SPACE

In this section we will discuss coordinate systems in three-dimensional space and some basic facts about surfaces in three dimensions.

1.1 RECTANGULAR COORDINATE SYSTEM

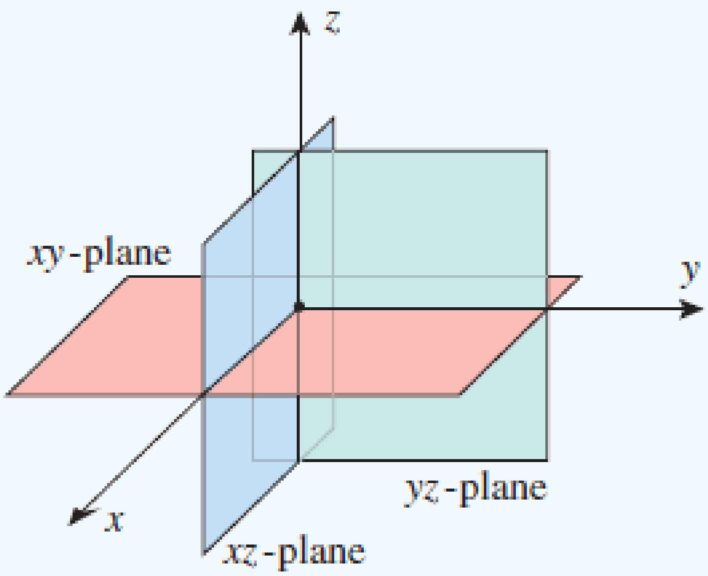
In the remainder of this unit we will call three-dimensional space **3-space**, two-dimensional space (a plane) **2-space**, and one-dimensional space (a line) **1-space**. Points in 3-space can be placed in one-to-one correspondence with triples of real numbers by using three mutually perpendicular coordinate lines, called the ***x-axis***, the ***y-axis***, and the ***z-axis***, (Figure 1.1) positioned so that their origins coincide. The three coordinate axes form a three-dimensional ***rectangular coordinate system*** (or ***Cartesian coordinate system***). The point of intersection of the coordinate axes is called the ***origin*** of the coordinate system.

Figure 1.1



The coordinate axes, taken in pairs, determine three ***coordinate planes***: the xy-plane, the xz-plane, and the yz-plane (Figure 1.2).

Figure 1.2



To each point P in 3-space we can assign a triple of real numbers by passing three planes through P parallel to the coordinate planes and letting a, b, and c be the coordinates of the intersections of those planes with the x-axis, y-axis, and z-axis, respectively (Figure 1.3). We call a, b, and c the x-coordinate, y-coordinate, and z-coordinate of P, respectively, and we denote the point P by (a, b, c) or by $P(a, b, c)$.

Figure 1.3

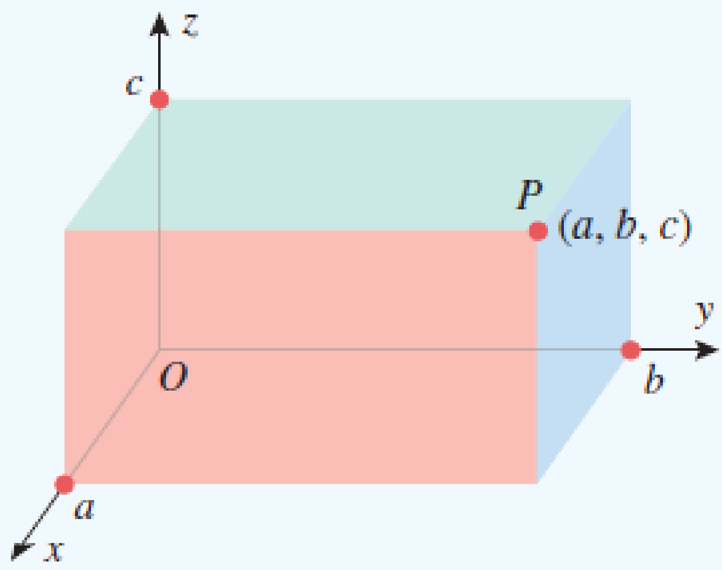
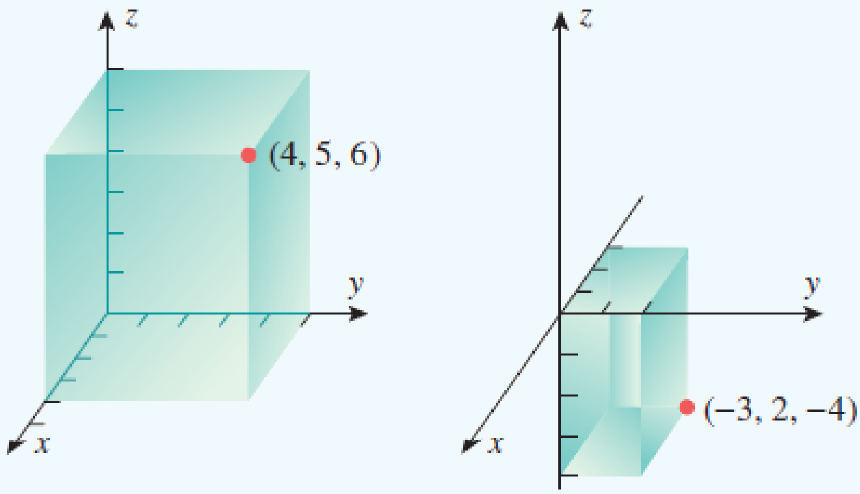


Figure 1.4 shows the points $(4, 5, 6)$ and $(-3, 2, -4)$.

Figure 1.4



The following are notable facts about three-dimensional rectangular coordinate systems:

REGION	DESCRIPTION
xy-plane	Consists of all points of the form $(x, y, 0)$
xz-plane	Consists of all points of the form $(x, 0, z)$
yz-plane	Consists of all points of the form $(0, y, z)$
x-axis	Consists of all points of the form $(x, 0, 0)$
y-axis	Consists of all points of the form $(0, y, 0)$
z-axis	Consists of all points of the form $(0, 0, z)$

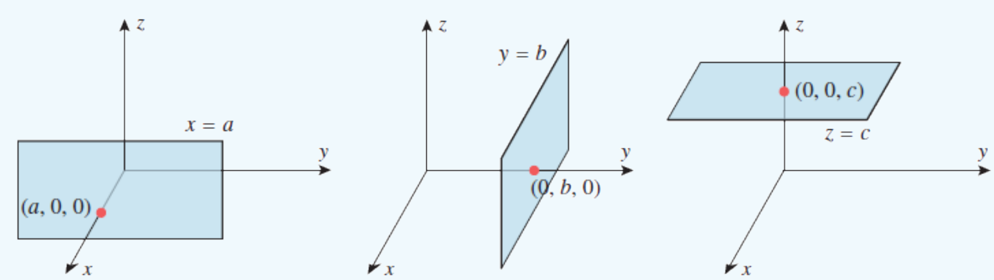
PLANES PARALLEL TO COORDINATE PLANES

Note that, at any point on the xy-plane, value of z-coordinate is 0. Now consider all points with $z = 1$, i.e. all points of the form $(x, y, 1)$. These points are seemed lie 1 unit above the xy-plane. So they lie on a plane parallel to xy-plane at which $z = 1$. Similarly, points of the form $(x, y, 2)$ also lie on a plane parallel to xy- plane, 2 units above it. In general if c is a constant, the points (x, y, c) lie on a plane parallel to xy-plane and the points (x, b, z) lie on a plane parallel to xz- plane, where b is a constant. We summarize these facts as,

POINTS	NATURE OF PLANE
$(x, y, 1), (x, y, -2.5), (x, y, k)$	parallel to xy-plane
$(x, 1, z), (x, -2.5, z), (x, k, z)$	parallel to xz-plane
$(1, y, z), (-2.5, y, z), (k, y, z)$	parallel to yz-plane

Here a, b, c are constants.

Figure 1.5



1.2 SPHERES

The distance between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{1}$$

The standard equation of the sphere in 3-space that has center (x_0, y_0, z_0) and radius r is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2 \tag{2}$$

If the terms in (2) are expanded and like terms are collected, then the resulting equation has the form

$$x^2 + y^2 + z^2 + Gx + Hy + Iz + J = 0 \tag{3}$$

which on completing the squares produces an equation of the form

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = k.$$

- If $k > 0$, then the graph of this equation is a sphere with center (x_0, y_0, z_0) and radius $\sqrt{k}$.
- If $k = 0$, then the sphere has radius zero, so the graph is the single point (x_0, y_0, z_0).
- If $k < 0$, the equation is not satisfied by any values of x, y, and z (why?), so it has no graph.

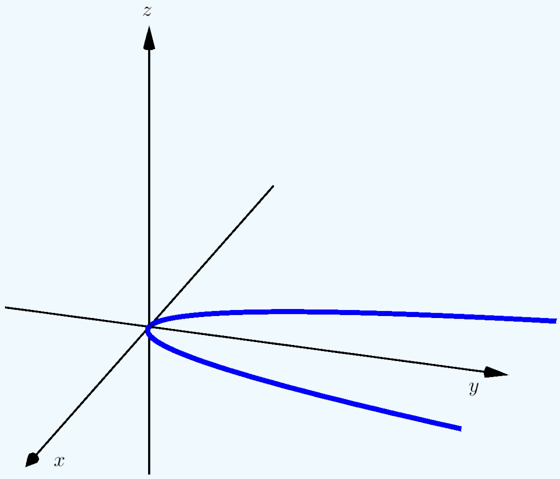
1.3 CYLINDRICAL SURFACES

We know that the graph of the equation

$$y = x^2$$

jeejamath@gmail.com

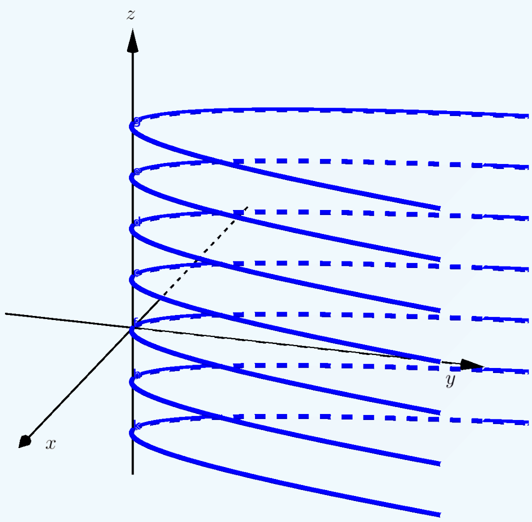
Figure 1.6



in an xy-coordinate system is a parabola. Now consider the same equation in 3-space. Here the equation does not contain the variable z. That means, whatever be the value of z, we have $y = x^2$. In other words, at any point on the graph of this equation in 3- space, the value of y is x^2, but any arbitrary value can be chosen for z and so for each value of z we get the parabola.

For example when $z = 1$, the graph is the parabola parallel to $y = x^2$ but 1 unit above the xy-plane. Similarly, when $z = -3$, the parabola will be 3 units below xy-plane.

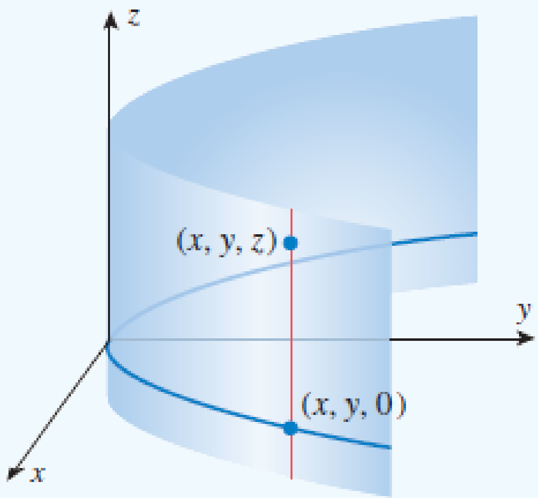
Figure 1.7



In general we get parabolas to the parabola in xy-plane, above and below it. That is

> *we can obtain the graph of $y = x^2$ in an xyz-coordinate system by first graphing the equation in the xy-plane and then translating that graph parallel to the z-axis to generate the entire graph.*

Figure 1.8



The process of generating a surface by translating a plane curve parallel to some line is called ***extrusion***, and surfaces that are generated by extrusion are called ***cylindrical surfaces***.

Theorem 1.1 (Cylindrical Surface)

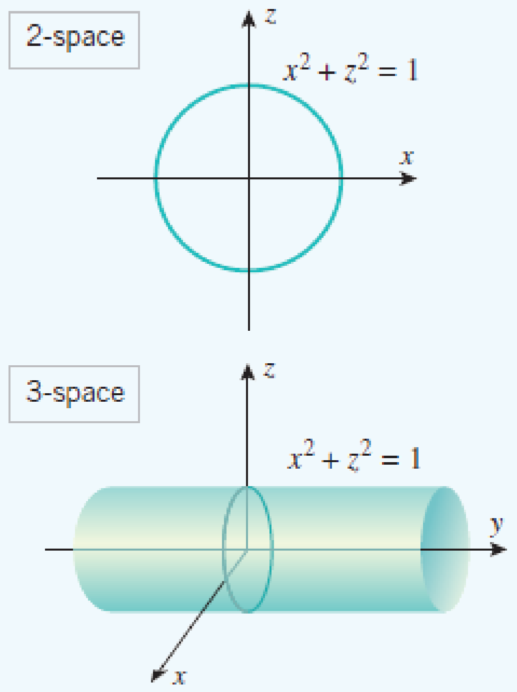
An equation that contains only two of the variables x, y, and z represents a cylindrical surface in an xyz-coordinate system. The surface can be obtained by graphing the equation in the coordinate plane of the two variables that appear in the equation and then translating that graph parallel to the axis of the missing variable.

Problem 1.1

Sketch the graph of $x^2 + z^2 = 1$ in 3-space.

Solution. Since y does not appear in this equation, the graph is a cylindrical surface generated by translating the plane curve $x^2 + z^2 = 1$ parallel to the y-axis. In the xz-plane the graph of the equation $x^2 + z^2 = 1$ is a circle. Thus, in 3-space the graph is a right circular cylinder along the y-axis.

Figure 1.9

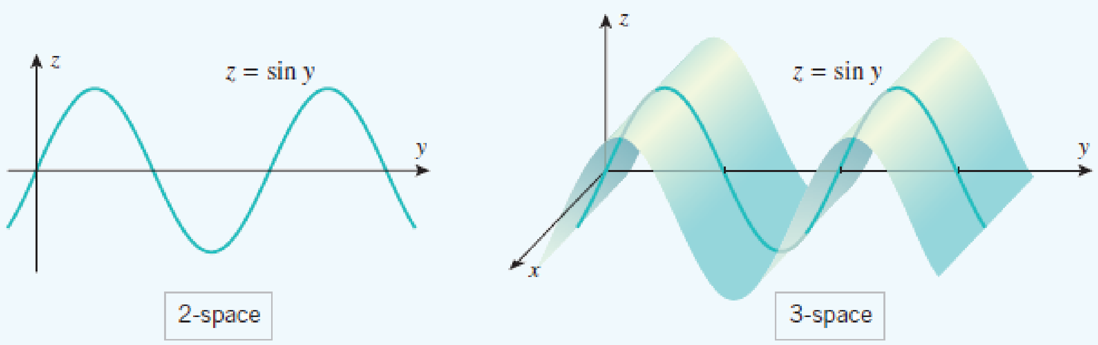


Problem 1.2

Sketch the graph of $z = \sin y$ in 3-space.

Solution. In 2-space, graph of $z = \sin y$ is the sine curve in yz-plane. In 3-space the graph is generated by translating this curve parallel to the x-axis, which is shown in Figure 1.10.

Figure 1.10

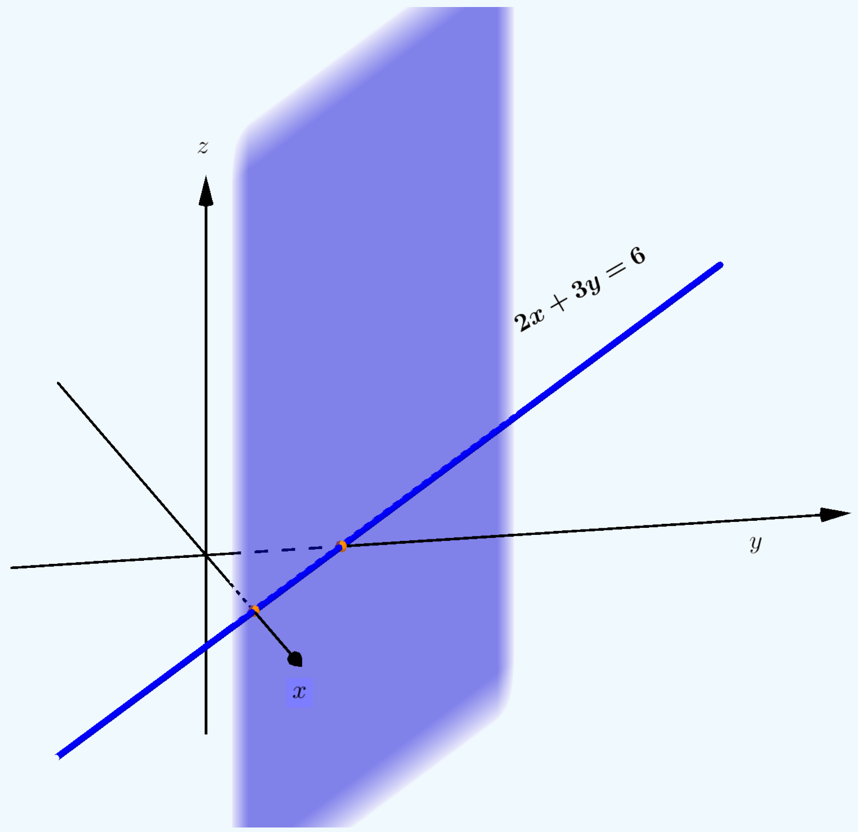


Problem 1.3

Sketch the graph of $2x + 3y = 6$ in 3-space.

Solution. In 2-space, graph of $2x + 3y = 6$ is the line in xy-plane passing through $(3, 0)$ and $(0, 2)$. In 3-space the graph is generated by translating this line parallel to the z-axis, which a plane.

Figure 1.11

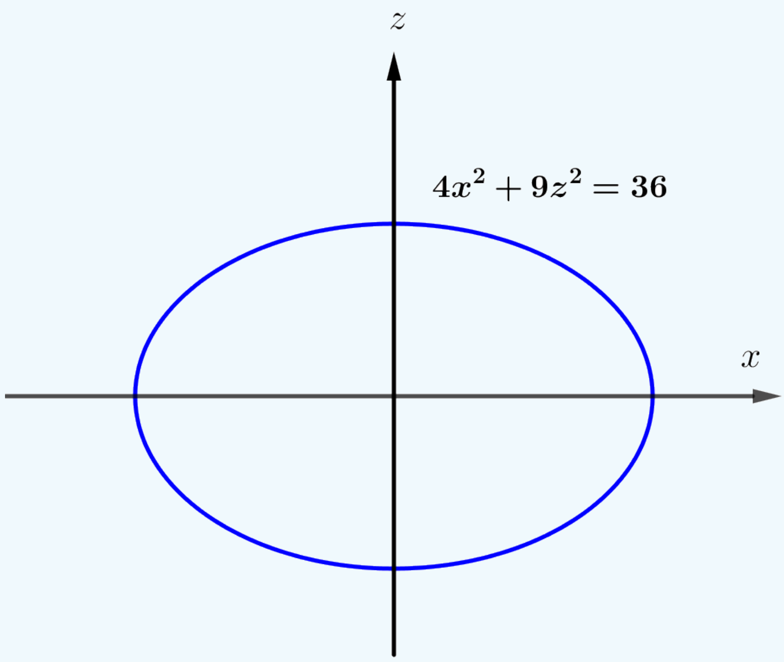


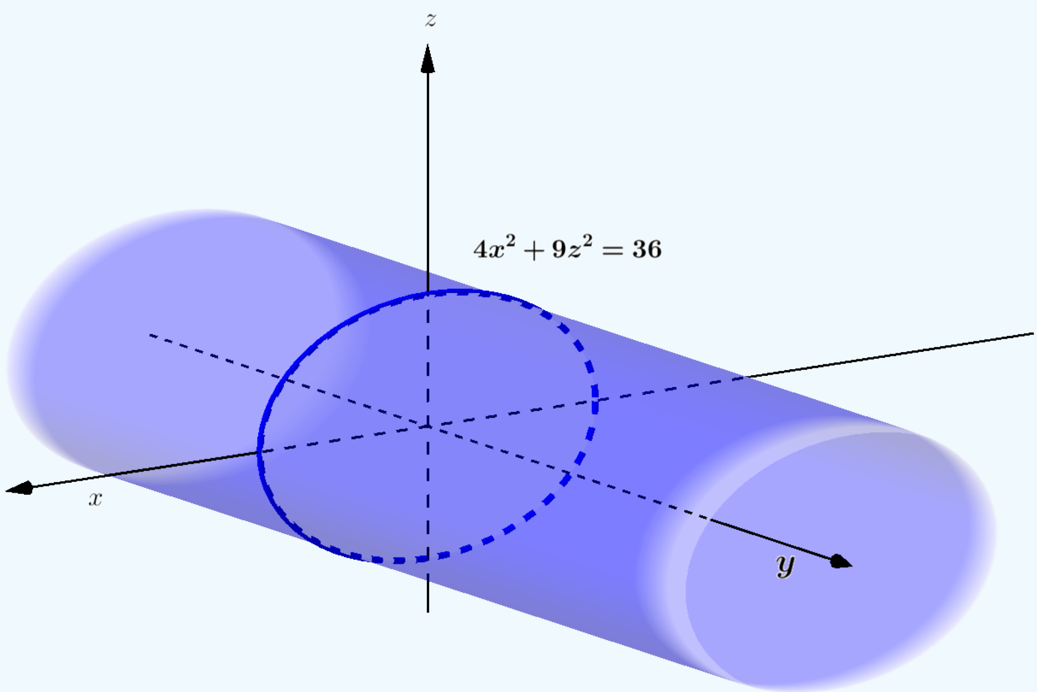
Problem 1.4

Sketch the graph of $4x^2 + 9z^2 = 36$ in 3-space.

Solution. In 2-space, graph of $4x^2 + 9z^2 = 36$ is an ellipse in xz-plane. In 3-space the graph is generated by translating this ellipse parallel to the y-axis, which is an elliptic cylinder.

Figure 1.12





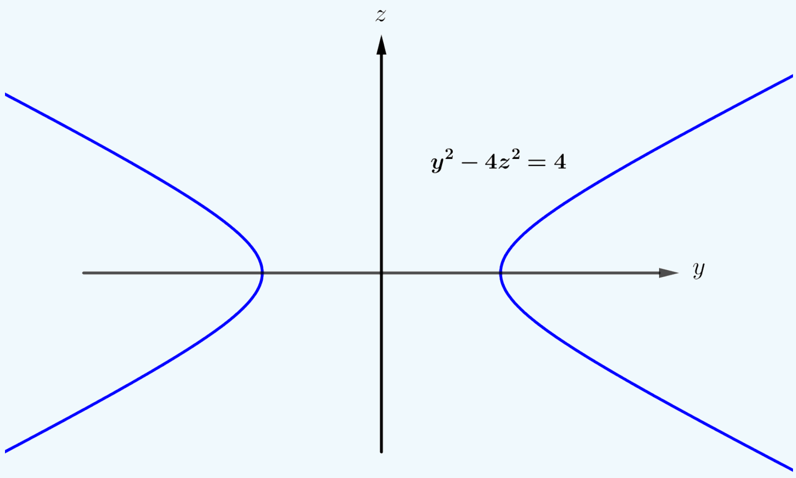
■

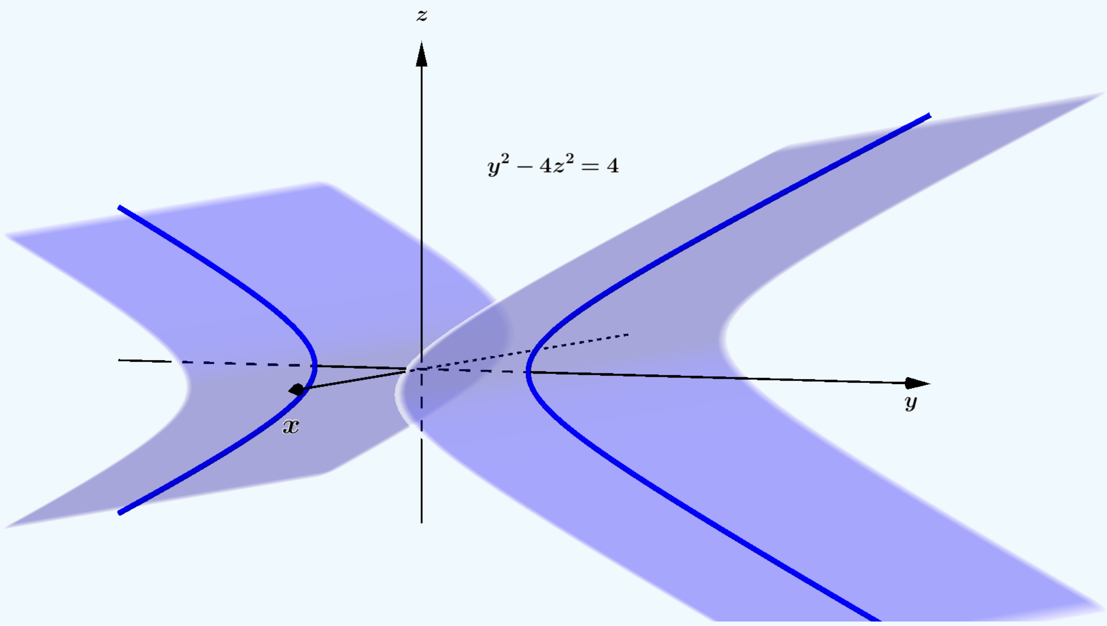
Problem 1.5

Sketch the graph of $y^2 - 4z^2 = 4$ in 3-space.

Solution. In 2-space, graph of $y^2 - 4z^2 = 4$ is a hyperbola in yz-plane. In 3-space the graph is generated by translating this hyperbola parallel to the x-axis, which is shown in Figure 1.13.

Figure 1.13

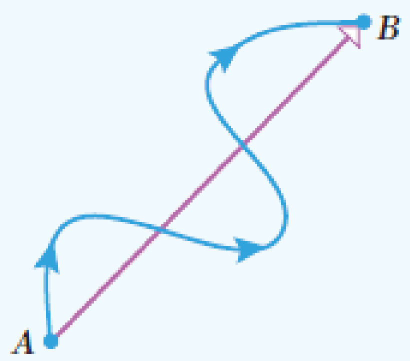




2 VECTORS

A particle that moves along a line can move in only two directions, so its direction of motion can be described by taking one direction to be positive and the other negative. Thus, the ***displacement*** or ***change in position*** of the point can be described by a signed real number. For example, a displacement of $3(=+3)$ describes a position change of 3 units in the positive direction, and a displacement of -3 describes a position change of 3 units in the negative direction. However, for a particle that moves in two dimensions or three dimensions, a plus or minus sign is no longer sufficient to specify the direction of motion-other methods are required. One method is to use an arrow, called a ***vector***, that points in the direction of motion and whose length represents the distance from the starting point to the ending point; this is called the ***displacement vector*** for the motion. For example, Figure 2.1 shows the displacement vector of a particle that moves from point A to point B along a circuitous path. Note that the length of the arrow describes the distance between the starting and ending points and not the actual distance travelled by the particle. Vectors can be used to describe any physical quantity that involves both a magnitude and a direction; forces and velocities are important examples.

Figure 2.1

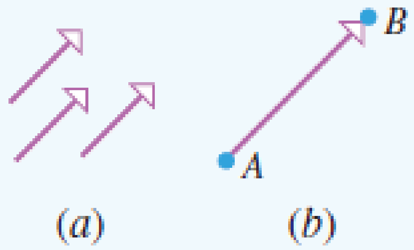


A displacement vector

VECTORS VIEWED GEOMETRICALLY

Vectors can be represented geometrically by arrows in 2-space or 3-space; the direction of the arrow specifies the direction of the vector, and the length of the arrow describes its magnitude. The tail of the arrow is called the ***initial point*** of the vector, and the tip of the arrow the ***terminal point***. We will denote vectors with lowercase boldface type such as $\mathbf{a}, \mathbf{k}, \mathbf{v}, \mathbf{w}$, and $\mathbf{x}$. When discussing vectors, we will refer to ***real numbers*** as ***scalars***. Scalars will be denoted by lowercase italic type such as a, k, v, w, and x. Two vectors, $\mathbf{v}$ and $\mathbf{w}$, are considered to be equal (also called equivalent) if they have the same length and same direction, in which case we write $\mathbf{v} = \mathbf{w}$. Geometrically, two vectors are equal if they are translations of one another; thus, the three vectors in Figure 2.2 (a) are equal, even though they are in different positions.

Figure 2.2



If the initial point of $\mathbf{v}$ is A and the terminal point is B, then we write $\mathbf{v} = \overrightarrow{AB}$ (see Figure 2.2 (b)). If the initial and terminal points of a vector coincide, then the vector has length zero; we call this the ***zero vector*** and denote it by $\mathbf{0}$.

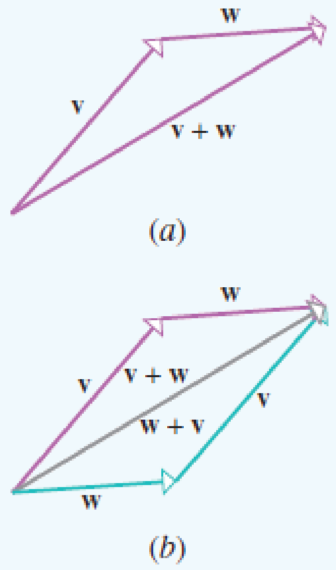
We have learned vectors and operations on vectors in higher secondary classes. We give a quick review of those topics here.

SUM OF VECTORS

If $\mathbf{v}$ and $\mathbf{w}$ are vectors, then the sum $\mathbf{v} + \mathbf{w}$ is the vector from the initial point of $\mathbf{v}$ to the terminal point of $\mathbf{w}$ when the vectors are positioned so the initial point of $\mathbf{w}$ is at the terminal point of $\mathbf{v}$.

If $\mathbf{v}$ and $\mathbf{w}$ are positioned so that they have the same initial point, the sum $\mathbf{v} + \mathbf{w}$ coincides with the diagonal of the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$.

Figure 2.3



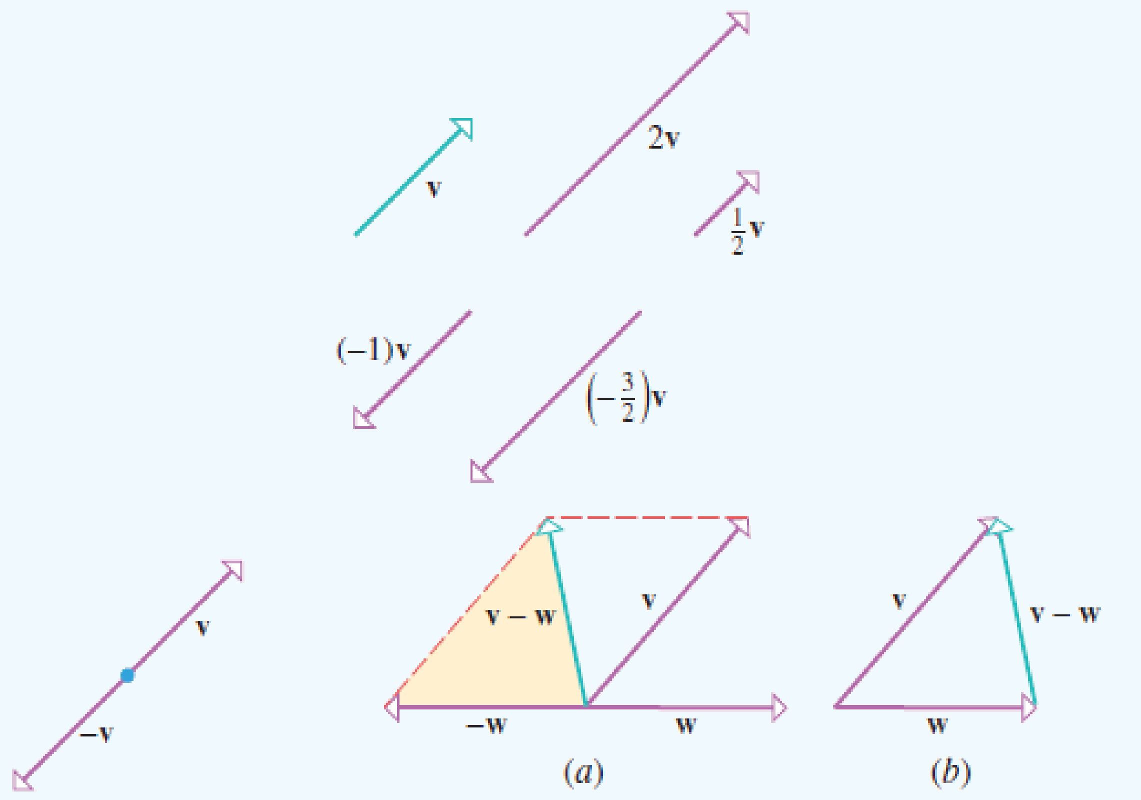
SCALAR MULTIPLE OF VECTORS

If $\mathbf{v}$ is a nonzero vector and k is a nonzero real number (a scalar), then the scalar multiple $k\mathbf{v}$ is defined to be the vector whose length is $|k|$ times the length of $\mathbf{v}$ and whose direction is the same as that of $\mathbf{v}$ if $k > 0$ and opposite to that of $\mathbf{v}$ if $k < 0$. We define $k\mathbf{v} = 0$ if $k = 0$ or $\mathbf{v} = \mathbf{0}$.

We say that $\mathbf{v}$ and $k\mathbf{v}$ are ***parallel vectors***. The vector $(-1)\mathbf{v}$ has the same length as $\mathbf{v}$ but is oppositely directed. We call $(-1)\mathbf{v}$ the negative of $\mathbf{v}$ and denote it by $-\mathbf{v}$. In particular, $-\mathbf{0} = (-1)\mathbf{0} = \mathbf{0}$. We also define

$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}).$$

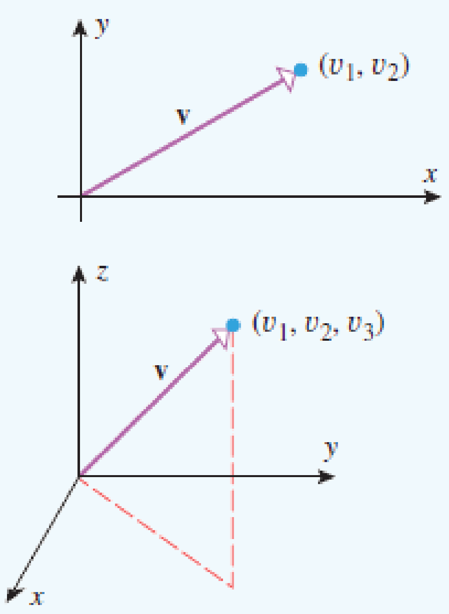
Figure 2.4



VECTORS IN COORDINATE SYSTEMS

If a vector **v** is positioned with its initial point at the origin of a rectangular coordinate system, then its terminal point will have coordinates of the form (v_1, v_2) or (v_1, v_2, v_3), depending on whether the vector is in 2-space or 3- space.

Figure 2.5



We call these coordinates the ***components*** of $\mathbf{v}$, and we write $\mathbf{v}$ in ***component form*** using the ***bracket notation***

$$\underbrace{\mathbf{v} = \langle v_1, v_2 \rangle}_{\boxed{\text{2-space}}} \quad \text{or} \quad \underbrace{\mathbf{v} = \langle v_1, v_2, v_3 \rangle}_{\boxed{\text{3-space}}} \tag{4}$$

In particular, the zero vectors in 2-space and 3-space are

$$\mathbf{0} = \langle 0, 0 \rangle \quad \text{and} \quad \mathbf{0} = \langle 0, 0, 0 \rangle$$

respectively.

Theorem 2.1

Two vectors are equivalent if and only if their corresponding components are equal. That is if $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, then $\mathbf{v} = \mathbf{w}$ if and only if $v_1 = w_1, v_2 = w_2$ and $v_3 = w_3$.

NORM OF A VECTOR

The distance between the initial and terminal points of a vector $\mathbf{v}$ is called the ***length***, the ***norm***, or the ***magnitude*** of $\mathbf{v}$ and is denoted by $\|\mathbf{v}\|$. If $\mathbf{v} = \langle v_1, v_2 \rangle$ is a vector in 2-space, then

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2} \tag{5}$$

and the norm of a vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ in 3-space is given by,

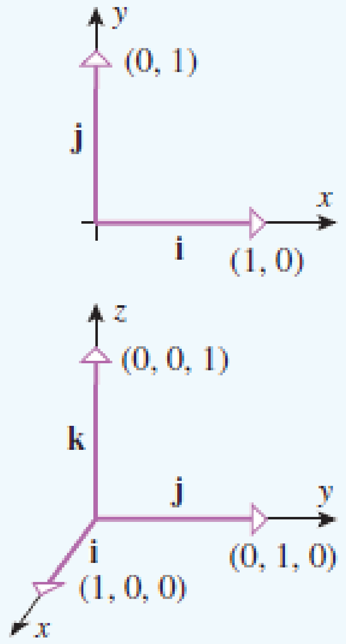
$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}. \tag{6}$$

UNIT VECTORS

A vector of length 1 is called a ***unit vector***. In an xy-coordinate system the unit vectors along the x-and y-axes are denoted by $\mathbf{i}$ and $\mathbf{j}$, respectively; and in an xyz-coordinate system the unit vectors along the x-, y-, and z-axes are denoted by $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$, respectively. Thus

$$\mathbf{i} = \langle 1, 0 \rangle, \qquad \mathbf{j} = \langle 0, 1 \rangle \qquad \boxed{\text{In 2-space}}$$
$$\mathbf{i} = \langle 1, 0, 0 \rangle, \qquad \mathbf{j} = \langle 0, 1, 0 \rangle, \qquad \mathbf{k} = \langle 0, 0, 1 \rangle. \qquad \boxed{\text{In 3-space}}$$

Figure 2.6



Using these unit vectors we represent vectors in 2-space and 3-space as

$$\begin{aligned}
\mathbf{v} &= \langle v_1, v_2 \rangle && \boxed{\text{2-space}} \\
&= \langle v_1, 0 \rangle + \langle 0, v_2 \rangle \\
&= v_1 \langle 1, 0 \rangle + v_2 \langle 0, 1 \rangle \\
&= v_1\mathbf{i} + v_2\mathbf{j} \\
\mathbf{v} &= \langle v_1, v_2, v_3 \rangle && \boxed{\text{3-space}} \\
&= \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle \\
&= v_1 \langle 1, 0, 0 \rangle + v_2 \langle 0, 1, 0 \rangle + v_3 \langle 0, 0, 1 \rangle \\
&= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.
\end{aligned}$$

If $\mathbf{v}$ is a non zero vector, then

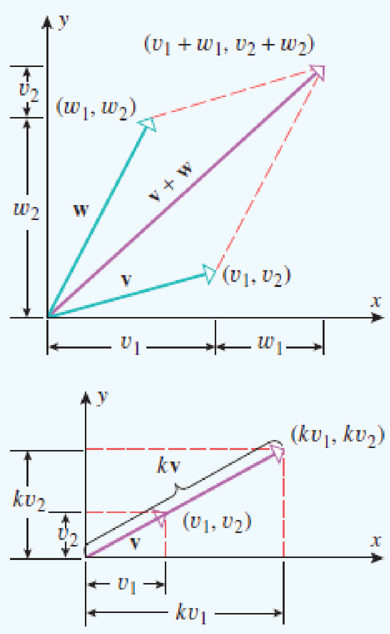
$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

is a unit vector in the direction of $\mathbf{v}$. The process of multiplying a vector $\mathbf{v}$ by the reciprocal of its length to obtain a unit vector with the same direction is called ***normalizing*** $\mathbf{v}$.

ARITHMETIC OPERATIONS ON VECTORS

The next theorem shows how to perform arithmetic operations on vectors using components.

Figure 2.7



Theorem 2.2

If $\mathbf{v} = \langle v_1, v_2 \rangle$ *and* $\mathbf{w} = \langle w_1, w_2 \rangle$ *are vectors in 2- space and* k *is any scalar, then*

- $\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2 \rangle$
- $\mathbf{v} - \mathbf{w} = \langle v_1 - w_1, v_2 - w_2 \rangle$
- $k\mathbf{v} = \langle kv_1, kv_2 \rangle$

Similarly, if $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ *are vectors in 3-space and* k *is any scalar, then*

- $\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$
- $\mathbf{v} - \mathbf{w} = \langle v_1 - w_1, v_2 - w_2, v_3 - w_3 \rangle$
- $k\mathbf{v} = \langle kv_1, kv_2, kv_3 \rangle$.

VECTORS WITH INITIAL POINT NOT AT THE ORIGIN

Suppose that $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points in 2-space, then the vector $\overrightarrow{P_1P_2}$ is given by

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \langle x_2, y_2 \rangle - \langle x_1, y_1 \rangle = \langle x_2 - x_1, y_2 - y_1 \rangle .$$

Similarly, if $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are points in 3-space, then the vector $\overrightarrow{P_1P_2}$ is given by

$$\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle .$$

This table provides some examples of vector notation in 2-space and 3-space.

2-SPACE	3-SPACE
$\langle 2,3\rangle = 2\mathbf{i}+3\mathbf{j}$	$\langle 2,-3,4\rangle = 2\mathbf{i}-3\mathbf{j}+4\mathbf{k}$
$\langle -4,0\rangle = -4\mathbf{i}+0\mathbf{j} = -4\mathbf{i}$	$\langle 0,3,0\rangle = 3\mathbf{j}$
$\langle 0,0\rangle = 0\mathbf{i}+0\mathbf{j} = \mathbf{0}$	$\langle 0,0,0\rangle = 0\mathbf{i}+0\mathbf{j}+0\mathbf{k} = \mathbf{0}$
$(3\mathbf{i}+2\mathbf{j})+(4\mathbf{i}+\mathbf{j}) = 7\mathbf{i}+3\mathbf{j}$	$(3\mathbf{i}+2\mathbf{j}-\mathbf{k})-(4\mathbf{i}-\mathbf{j}+2\mathbf{k}) = -\mathbf{i}+3\mathbf{j}-3\mathbf{k}$
$5(6\mathbf{i}-2\mathbf{j}) = 30\mathbf{i}-10\mathbf{j}$	$2(\mathbf{i}+\mathbf{j}-\mathbf{k})+4(\mathbf{i}-\mathbf{j}) = 6\mathbf{i}-2\mathbf{j}-2\mathbf{k}$
$\\|2\mathbf{i}-3\mathbf{j}\\| = \sqrt{2^2+(-3)^2} = \sqrt{13}$	$\\|\mathbf{i}+2\mathbf{j}-3\mathbf{k}\\| = \sqrt{1^2+2^2+(-3)^2} = \sqrt{14}$
$\\|v_1\mathbf{i}+v_2\mathbf{j}\\| = \sqrt{v_1^2+v_2^2}$	$\\|\langle v_1,v_2,v_3\rangle\\| = \sqrt{v_1^2+v_2^2+v_3^2}$

VECTORS DETERMINED BY LENGTH AND ANGLE

If $\mathbf{v}$ is a nonzero vector with its initial point at the origin of an xy- coordinate system, and if θ is the angle from the positive x-axis to the radial line through $\mathbf{v}$, then $\mathbf{v}$ can be expressed in trigonometric form as

$$\mathbf{v} = \|\mathbf{v}\| \langle \cos\theta, \sin\theta\rangle \qquad \text{or} \qquad \mathbf{v} = \|\mathbf{v}\|\cos\theta\mathbf{i} + \|\mathbf{v}\|\sin\theta\mathbf{j} \tag{7}$$

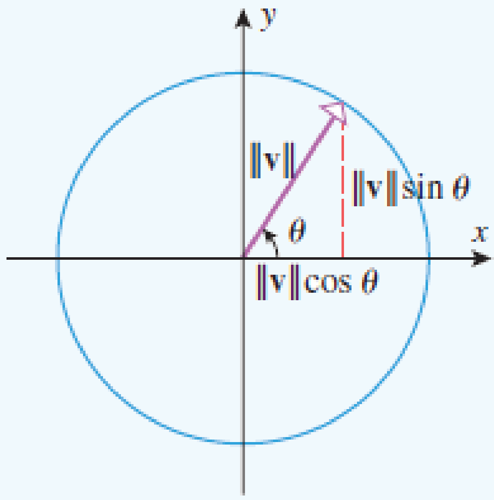
In this case, is given by

$$x - \text{component} = \|\mathbf{v}\|\cos\theta$$
$$y - \text{component} = \|\mathbf{v}\|\sin\theta.$$

In particular, if $\mathbf{u}$ is a unit vector, then

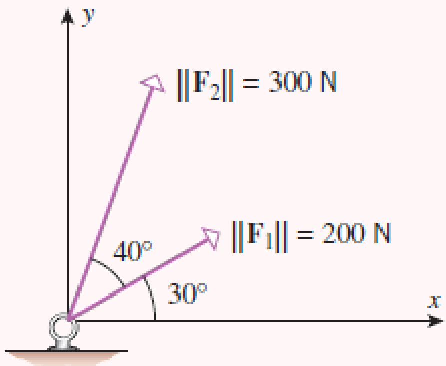
$$\mathbf{u} = \langle \cos\theta, \sin\theta\rangle \qquad \text{or} \qquad \mathbf{v} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}. \tag{8}$$

Figure 2.8



Problem 2.1

Suppose that two forces are applied to an eye bracket, as shown in the figure. Find the magnitude of the resultant and the angle θ that it makes with the positive x-axis.



Solution. Note that $\mathbf{F}_1$ makes an angle of $30°$ with the positive x-axis and $\mathbf{F}_2$ makes an angle of $30° + 40° = 70°$ with the positive x-axis. Since we are given that $\|\mathbf{F}_1\| = 200\,N$ and $\|\mathbf{F}_2\| = 300\,N$, we have

$$\mathbf{F}_1 = 200\,\langle\cos 30°, \sin 30°\rangle = 200\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle = \left\langle 100\sqrt{3}, 100\right\rangle$$

and

$$\mathbf{F}_2 = 300\,\langle\cos 70°, \sin 70°\rangle = \langle 300\cos 70°, 300\sin 70°\rangle$$

Therefore, the resultant $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ is given by

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2\\ &= \left\langle 100\sqrt{3} + 300\cos 70^\circ, 100 + 300\sin 70^\circ \right\rangle\\ &= 100\left\langle \sqrt{3} + 3\cos 70^\circ, 1 + 3\sin 70^\circ \right\rangle\\ &\approx \langle 275.8, 381.9\rangle\end{aligned}$$

The magnitude of the resultant is

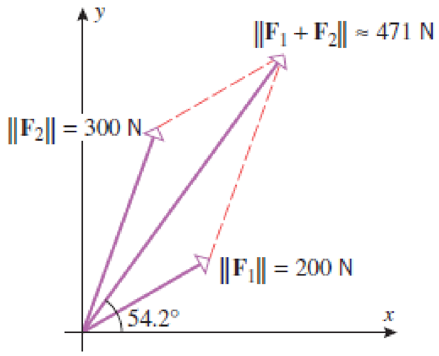
$$\|\mathbf{F}\| = \sqrt{275.8^2 + 381.9^2} \approx 471\,N.$$

Let θ be the angle $\mathbf{F}$ makes with the positive x-axis when the initial point of $\mathbf{F}$ is at the origin. From the x-component of $\mathbf{F}$, we have

$$\|\mathbf{F}\|\cos\theta = 100\sqrt{3} + 300\cos 70^\circ \Rightarrow \cos\theta = \frac{100\sqrt{3} + 300\cos 70^\circ}{\|\mathbf{F}\|}$$

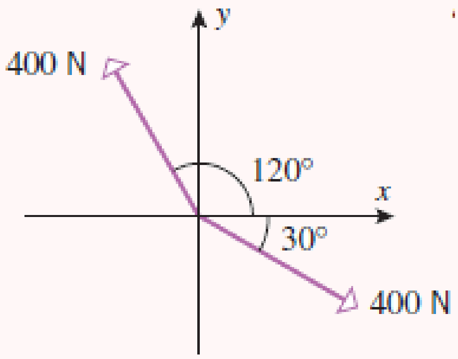
Since the terminal point of $\mathbf{F}$ is in the first quadrant, we have

$$\theta = \cos^{-1}\left(\frac{100\sqrt{3} + 300\cos 70^\circ}{\|\mathbf{F}\|}\right) \approx 54.2^\circ$$



Problem 2.2

Find the magnitude of the resultant force and the angle that it makes with the positive x-axis.



Solution. From the figure, we have

$$\begin{aligned}
\mathbf{F}_1 &= 400\,\langle\cos(-30^\circ),\sin(-30^\circ)\rangle\\
&= 400\,\langle\cos 30^\circ,-\sin 30^\circ\rangle\\
&= 400\left\langle\frac{\sqrt{3}}{2},-\frac{1}{2}\right\rangle\\
&= 200\left\langle\sqrt{3},-1\right\rangle\\
\mathbf{F}_2 &= 400\,\langle\cos 120^\circ,\sin 120^\circ\rangle\\
&= 400\,\langle-\sin 30^\circ,\cos 30^\circ\rangle\\
&= 400\left\langle-\frac{1}{2},\frac{\sqrt{3}}{2}\right\rangle\\
&= 200\left\langle-1,\sqrt{3}\right\rangle.
\end{aligned}$$

Hence the resultant $\mathbf{F}$ is given by

$$\begin{aligned}
\mathbf{F} &= \mathbf{F}_1+\mathbf{F}_2\\
&= 200\left\langle\sqrt{3},-1\right\rangle+200\left\langle-1,\sqrt{3}\right\rangle\\
&= 200\left\langle\sqrt{3}-1,\sqrt{3}-1\right\rangle.
\end{aligned}$$

The magnitude of the resultant is

$$\|\mathbf{F}\| = \sqrt{200^2(\sqrt{3}-1)^2+200^2(\sqrt{3}-1)^2} = 2\sqrt{2\times 200^2(\sqrt{3}-1)^2} = 200(\sqrt{3}-1)\sqrt{2}\approx 207\,N.$$

Let θ be the angle $\mathbf{F}$ makes with the positive x-axis when the initial point of $\mathbf{F}$ is at the origin. Clearly the terminal point of $\mathbf{F}$ is in the first quadrant. Also the x and y components are same for $\mathbf{F}$. Hence

$$\theta = 45^\circ.$$

2.1 DOT PRODUCT

If $\mathbf{u}=\langle u_1,u_2\rangle$ and $\mathbf{v}=\langle v_1,v_2\rangle$ are vectors in 2- space, then the dot product of $\mathbf{u}$ and $\mathbf{v}$ is written as $\mathbf{u}\cdot\mathbf{v}$ and is defined as

$$\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2.$$

Similarly, if $\mathbf{u}=\langle u_1,u_2,u_3\rangle$ and $\mathbf{v}=\langle v_1,v_2,v_3\rangle$ are vectors in 3-space, then their dot product is defined as

$$\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2+u_3v_3.$$

► Note that dot product of two vectors is a scalar.
The following are some important properties of dot product

Theorem 2.3

If $\mathbf{u}, \mathbf{v}$, *and* $\mathbf{w}$ *are vectors in 2- or 3-space and* k *is a scalar, then:*

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

(c) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$

(d) $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

(e) $\mathbf{0} \cdot \mathbf{v} = 0$

(f) $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$

(g) $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$

Proof. We assume that $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, then,

(a) $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = v_1u_1 + v_2u_2 + v_3u_3 = \mathbf{v} \cdot \mathbf{u}$

(b) We have

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \langle u_1, u_2, u_3 \rangle \cdot (\langle v_1, v_2, v_3 \rangle + \langle w_1, w_2, w_3 \rangle) \\
&= \langle u_1, u_2, u_3 \rangle \cdot \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle \\
&= u_1(v_1 + w_1) + u_2(v_2 + w_2) + u_3(v_3 + w_3) \\
&= (u_1v_1 + u_2v_2 + u_3v_3) + (u_1w_1 + u_2w_2 + u_3w_3) \\
&= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}.
\end{aligned}$$

(c) Now,

$$\begin{aligned}
k(\mathbf{u} \cdot \mathbf{v}) &= k(u_1v_1 + u_2v_2 + u_3v_3) \\
&= (ku_1)v_1 + (ku_2)v_2 + (ku_3)v_3 \\
&= (k\mathbf{u}) \cdot \mathbf{v}
\end{aligned}$$

Similarly

$$\begin{aligned}
k(\mathbf{u} \cdot \mathbf{v}) &= k(u_1v_1 + u_2v_2 + u_3v_3) \\
&= u_1(kv_1) + u_2(kv_2) + u_3(kv_3) \\
&= \mathbf{u} \cdot (k\mathbf{v}).
\end{aligned}$$

(d) Clearly

$$\mathbf{v} \cdot \mathbf{v} = v_1v_1 + v_2v_2 + v_3v_3 = v_1^2 + v_2^2 + v_3^2 = \|\mathbf{v}\|^2.$$

(e) Since $\mathbf{i} = \langle 1, 0, 0 \rangle$, from (d), we have

$$\mathbf{i} \cdot \mathbf{i} = \|\mathbf{i}\|^2 = 1^2 + 0^2 + 0^2 = 1.$$

Similarly

$$\mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1.$$

(f) Since $\mathbf{i} = \langle 1, 0, 0 \rangle, \mathbf{j} = \langle 0, 1, 0 \rangle$ and $\mathbf{k} = \langle 0, 0, 1 \rangle$, we have

$$\mathbf{i} \cdot \mathbf{j} = 1 \times 0 + 0 \times 1 + 0 \times 0 = 0.$$

Similarly

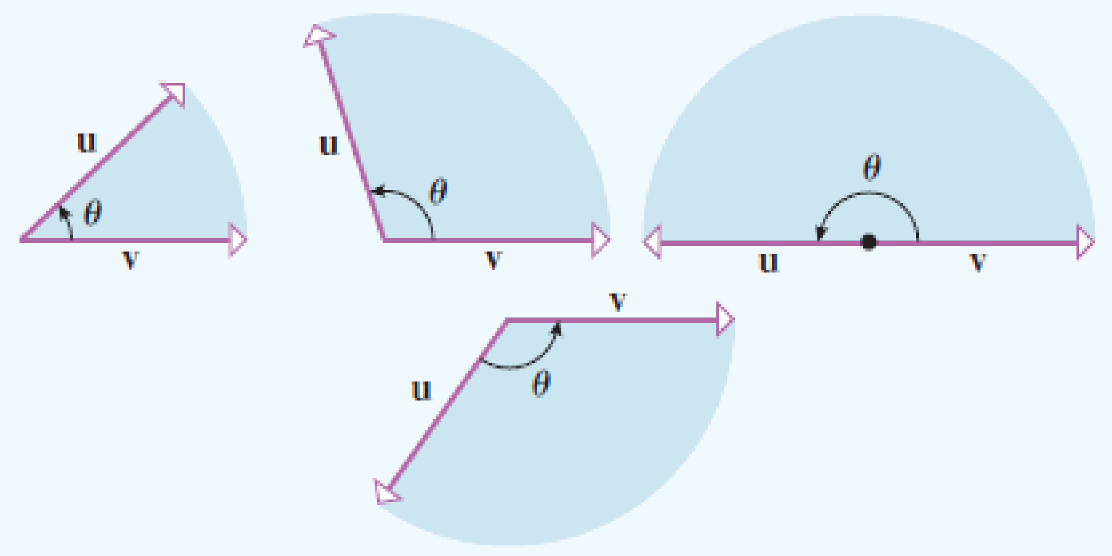
$$\mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0.$$

From Theorem 2.3(d), it is clear that

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}. \tag{9}$$

ANGLE BETWEEN VECTORS

Figure 2.9



Suppose that $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in 2-space or 3-space that are positioned so their initial points coincide. We define the angle between $\mathbf{u}$ and $\mathbf{v}$ to be the angle θ determined by the vectors that satisfies the condition $0 \le \theta \le \pi$.

Theorem 2.4

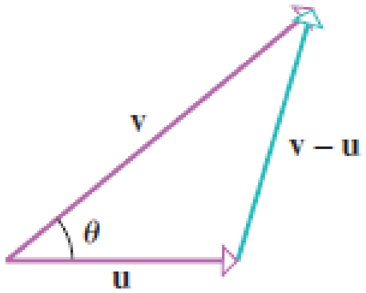
If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in 2-space or 3-space and if θ is the angle between them, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} \tag{10}$$

Proof. Suppose that the vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{v} - \mathbf{u}$ are positioned to form three sides of a triangle (see the figure below). It follows from the law of cosines that

$$\|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{u}\|^2 - 2\,\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta. \tag{11}$$

Using the properties of dot product,



$$\begin{aligned}\|\mathbf{v} - \mathbf{u}\|^2 &= (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u}) \\ &= \mathbf{v} \cdot (\mathbf{v} - \mathbf{u}) - \mathbf{u} \cdot (\mathbf{v} - \mathbf{u}) \\ &= \mathbf{v} \cdot \mathbf{v} - (\mathbf{v} \cdot \mathbf{u}) - (\mathbf{u} \cdot \mathbf{v}) + \mathbf{u} \cdot \mathbf{u} \\ &= \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v}) - (\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{u}\|^2 \\ &= \|\mathbf{v}\|^2 - 2(\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{u}\|^2\end{aligned}$$

Substituting to (11), we get

$$\|\mathbf{u}\|^2 + \|\mathbf{u}\|^2 - 2\,\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta = \|\mathbf{v}\|^2 - 2(\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{u}\|^2$$

which shows that

$$\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta = \mathbf{u} \cdot \mathbf{v}.$$

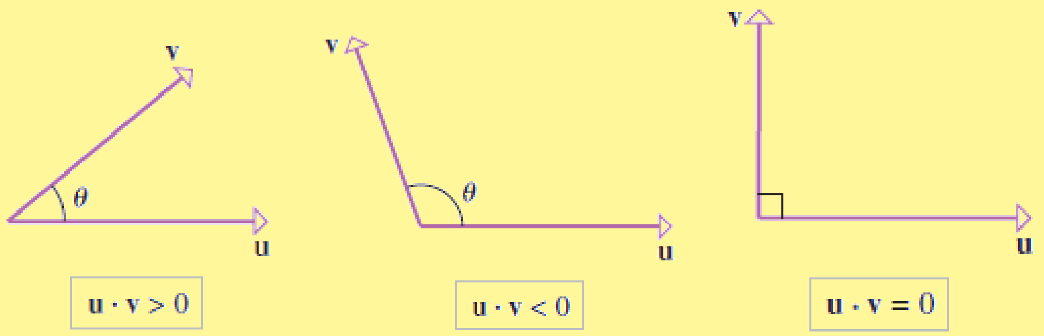
Hence

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$$ ■

It is evident from (10) that the sign of dot product is same as the sign of $\cos\theta$ which makes us easier to determine the nature of angle from the sign of dot product. Thus we have

Suppose $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors and let θ be the angle between them. If

- $\mathbf{u} \cdot \mathbf{v} > 0$, then $0 \le \theta < \frac{\pi}{2}$ (θ is acute),
- $\mathbf{u} \cdot \mathbf{v} < 0$, then $\frac{\pi}{2} \le \theta < \pi$, ($\theta$ is obtuse)
- $\mathbf{u} \cdot \mathbf{v} = 0$, then $\theta = \frac{\pi}{2}$, i.e. the vectors are perpendicular or *orthogonal.*



Problem 2.3

In each part, determine whether $\mathbf{u}$ and $\mathbf{v}$ make an acute angle, an obtuse angle, or are orthogonal.

(a) $\mathbf{u} = 7\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}, \mathbf{v} = -8\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

(b) $\mathbf{u} = 6\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \mathbf{v} = 4\mathbf{i} - 6\mathbf{k}$

(c) $\mathbf{u} = \langle 1, 1, 1 \rangle, \mathbf{v} = \langle -1, 0, 0 \rangle$

(d) $\mathbf{u} = \langle 4, 1, 6 \rangle, \mathbf{v} = \langle -3, 0, 2 \rangle$

Solution. (a) We have

$$\mathbf{u} \cdot \mathbf{v} = (7\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) \cdot (-8\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) = 7 \times (-8) + 3 \times 4 + 5 \times 2 = -44 < 0$$

Hence $\mathbf{u}$ and $\mathbf{v}$ make an obtuse angle.

(b) We have

$$\mathbf{u} \cdot \mathbf{v} = (6\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \cdot (4\mathbf{i} - 6\mathbf{k}) = 6 \times 4 + 1 \times 0 + 3 \times (-6) = 6 > 0$$

Hence $\mathbf{u}$ and $\mathbf{v}$ make an acute angle.

(c) We have

$$\mathbf{u} \cdot \mathbf{v} = \langle 1, 1, 1 \rangle \cdot \langle -1, 0, 0 \rangle = 1 \times (-1) + 1 \times 0 + 1 \times 0 = -1 < 0$$

Hence $\mathbf{u}$ and $\mathbf{v}$ make an obtuse angle.

(d) We have

$$\mathbf{u} \cdot \mathbf{v} = \langle 4, 1, 6 \rangle \cdot \langle -3, 0, 2 \rangle = 4 \times (-3) + 1 \times 0 + 6 \times 2 = 0$$

Hence the vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

■

DIRECTION ANGLES

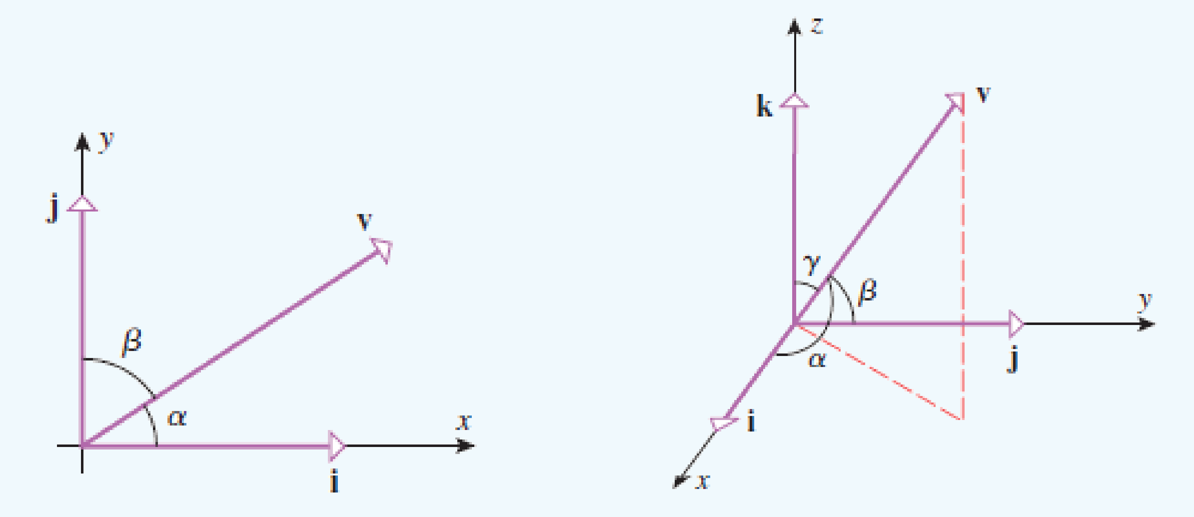
In an xy-coordinate system, the direction of a nonzero vector $\mathbf{v}$ is completely determined by the angles α and β between $\mathbf{v}$ and the unit vectors $\mathbf{i}$ and $\mathbf{j}$, and in an xyz-coordinate system the direction is completely determined by the angles α, β, and γ between $\mathbf{v}$ and the unit vectors $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$. These angles are called the ***direction angles*** of $\mathbf{v}$ and cosine of these angles are called the ***direction cosines*** of $\mathbf{v}$.

If $\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\begin{aligned}\cos\alpha &= \frac{\mathbf{v}\cdot\mathbf{i}}{\|\mathbf{v}\|\,\|\mathbf{i}\|}\\ &= \frac{(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})\cdot\mathbf{i}}{\|\mathbf{v}\|\times 1}\\ &= \frac{v_1}{\|\mathbf{v}\|}\end{aligned}$$

$$\begin{aligned}\cos\beta &= \frac{\mathbf{v}\cdot\mathbf{j}}{\|\mathbf{v}\|\,\|\mathbf{j}\|}\\ &= \frac{(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})\cdot\mathbf{j}}{\|\mathbf{v}\|\times 1}\\ &= \frac{v_2}{\|\mathbf{v}\|}\\ \cos\gamma &= \frac{\mathbf{v}\cdot\mathbf{k}}{\|\mathbf{v}\|\,\|\mathbf{k}\|}\\ &= \frac{(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})\cdot\mathbf{k}}{\|\mathbf{v}\|\times 1}\\ &= \frac{v_3}{\|\mathbf{v}\|}\end{aligned}$$

Figure 2.10



Theorem 2.5 (Direction Cosines)

The direction cosines of a nonzero vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ *are*

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|}, \quad \cos\beta = \frac{v_2}{\|\mathbf{v}\|}, \quad \cos\gamma = \frac{v_3}{\|\mathbf{v}\|}. \tag{12}$$

▶ Clearly direction cosines of a nonzero vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ are components of the unit vector $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$.

Problem 2.4

Show that the direction cosines of a nonzero vector $\mathbf{v}$, satisfy the equation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \tag{13}$$

Solution. Let α, β, and γ be the direction angles of $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|}, \quad \cos\beta = \frac{v_2}{\|\mathbf{v}\|}, \quad \text{and } \cos\gamma = \frac{v_3}{\|\mathbf{v}\|}.$$

Then

$$\begin{aligned}\cos^2\alpha + \cos^2\beta + \cos^2\gamma &= \left(\frac{v_1}{\|\mathbf{v}\|}\right)^2 + \left(\frac{v_2}{\|\mathbf{v}\|}\right)^2 + \left(\frac{v_3}{\|\mathbf{v}\|}\right)^2 \\ &= \frac{v_1^2 + v_2^2 + v_3^2}{\|\mathbf{v}\|^2} \\ &= \frac{\|\mathbf{v}\|^2}{\|\mathbf{v}\|^2} \\ &= 1.\end{aligned}$$

■

Problem 2.5

Find the direction cosines of the vector $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$, and approximate the direction angles to the nearest degree.

Solution. From the vector, $v_1 = 2, v_2 = -4, v_3 = 4$. We have $\|\mathbf{v}\| = \sqrt{2^2 + (-4)^2 + 4^2} = \sqrt{4 + 16 + 16} = \sqrt{36} = 6$. The directin cosines are given by

$$\begin{aligned}\cos\alpha &= \frac{v_1}{\|\mathbf{v}\|} = \frac{2}{6} = \frac{1}{3} \\ \cos\beta &= \frac{v_2}{\|\mathbf{v}\|} = \frac{-4}{6} = -\frac{2}{3} \\ \cos\gamma &= \frac{v_3}{\|\mathbf{v}\|} = \frac{4}{6} = \frac{2}{3}.\end{aligned}$$

Then

$$\begin{aligned}\alpha &= \cos^{-1}\left(\frac{1}{3}\right) \approx 71^\circ \\ \beta &= \cos^{-1}\left(-\frac{2}{3}\right) \approx 132^\circ \\ \gamma &= \cos^{-1}\left(\frac{2}{3}\right) \approx 48^\circ.\end{aligned}$$

■

Problem 2.6

Find the direction cosines of $\mathbf{v}$ and confirm that they satisfy Equation (12). Then use the direction cosines to approximate the direction angles to the nearest degree.

(a) $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$

(b) $\mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

(c) $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$

(d) $\mathbf{v} = 3\mathbf{i} - 4\mathbf{k}$

Solution. (a) We have

$$\|\mathbf{v}\| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{3}$$

The direction cosines are given by

$$\cos\alpha = \frac{1}{\sqrt{3}}, \quad \cos\beta = \frac{1}{\sqrt{3}}, \quad \cos\gamma = -\frac{1}{\sqrt{3}}.$$

Then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1.$$

Also

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.74^\circ$$
$$\beta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.74^\circ$$
$$\gamma = \cos^{-1}\left(-\frac{1}{\sqrt{3}}\right) \approx 125.26^\circ.$$

(b) We have

$$\|\mathbf{v}\| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3.$$

The direction cosines are given by

$$\cos\alpha = \frac{2}{3}, \quad \cos\beta = -\frac{2}{3}, \quad \cos\gamma = \frac{1}{3}.$$

Then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1.$$

Also

$$\alpha = \cos^{-1}\left(\frac{2}{3}\right) \approx 48.19^\circ$$
$$\beta = \cos^{-1}\left(-\frac{2}{3}\right) \approx 131.81^\circ$$
$$\gamma = \cos^{-1}\left(\frac{1}{3}\right) \approx 70.53^\circ.$$

(c) We have

$$\|\mathbf{v}\| = \sqrt{3^2 + (-2)^2 + (-6)^2} = \sqrt{49} = 7.$$

The direction cosines are given by

$$\cos\alpha = \frac{3}{7}, \quad \cos\beta = -\frac{2}{7}, \quad \cos\gamma = -\frac{6}{7}.$$

Then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{3}{7}\right)^2 + \left(-\frac{2}{7}\right)^2 + \left(-\frac{6}{7}\right)^2 = \frac{9}{49} + \frac{4}{49} + \frac{36}{49} = 1.$$

Also

$$\begin{aligned} \alpha &= \cos^{-1}\left(\frac{3}{7}\right) \approx 64.62^\circ \\ \beta &= \cos^{-1}\left(-\frac{2}{7}\right) \approx 106.6^\circ \\ \gamma &= \cos^{-1}\left(-\frac{6}{7}\right) \approx 149^\circ. \end{aligned}$$

(d) We have

$$\|\mathbf{v}\| = \sqrt{3^2 + 0^2 + (-4)^2} = \sqrt{25} = 5.$$

The direction cosines are given by

$$\cos\alpha = \frac{3}{5}, \quad \cos\beta = \frac{0}{5} = 0, \quad \cos\gamma = -\frac{4}{5}.$$

Then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{3}{5}\right)^2 + 0^2 + \left(-\frac{4}{5}\right)^2 = \frac{9}{25} + 0 + \frac{16}{25} = 1.$$

Also

$$\begin{aligned} \alpha &= \cos^{-1}\left(\frac{3}{5}\right) \approx 53.13^\circ \\ \beta &= \cos^{-1} 0 = 90^\circ \\ \gamma &= \cos^{-1}\left(-\frac{4}{5}\right) \approx 143.13^\circ. \end{aligned}$$

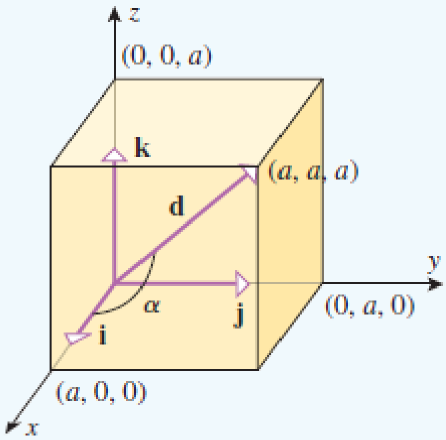
■

Problem 2.7

Find the angle between a diagonal of a cube and one of its edges.

Solution. Assume that the cube has side a, and introduce a coordinate system with one vertex as the origin and the sides intersecting at that vertex as x, y and z axes, as shown in Figure 2.11.

Figure 2.11



In this coordinate system the vector

$$\mathbf{d} = a\mathbf{i} + a\mathbf{j} + a\mathbf{k}$$

is a diagonal of the cube and the unit vectors $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ run along the edges. By symmetry, the diagonal makes the same angle with each edge, so it is sufficient to find the angle between $\mathbf{d}$ and $\mathbf{i}$ (the direction angle α). Thus,

$$\begin{aligned}\cos\alpha &= \frac{\mathbf{d}\cdot\mathbf{i}}{\|\mathbf{d}\|\,\|\mathbf{i}\|}\\ &= \frac{(a\mathbf{i} + a\mathbf{j} + a\mathbf{k})\cdot\mathbf{i}}{\sqrt{a^2+a^2+a^2}\times 1}\\ &= \frac{a}{\sqrt{3a^2}}\\ &= \frac{a}{\sqrt{3}a}\\ &= \frac{1}{\sqrt{3}}\end{aligned}$$

and hence

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.7^\circ.$$

DECOMPOSING VECTORS INTO ORTHOGONAL COMPONENTS

We know that any vector $\mathbf{v} = \langle v_1, v_2\rangle$ can be represented as

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j},$$

here $\mathbf{i}$ and $\mathbf{j}$ are orthogonal unit vectors along x and y axes. If we let $\mathbf{w}_1 = v_1\mathbf{i}$ and $\mathbf{w}_2 = v_2\mathbf{j}$, then $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal vectors and

$$\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2.$$

Hence any vector in 2-space is a sum of two *orthogonal vectors* along x and y axes.

We now show that if we are given two orthogonal unit vector $\mathbf{e}_1$ and $\mathbf{e}_2$ in 2-space, then any vector $\mathbf{v}$ in 2-space can be expressed as a sum

$$\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$$

where $\mathbf{w}_1$ is along $\mathbf{e}_1$ and $\mathbf{w}_2$ is along $\mathbf{e}_2$ so that $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal vectors. Then

$$\mathbf{w}_1 = k_1\mathbf{e}_1 \quad \text{and} \quad \mathbf{w}_2 = k_2\mathbf{e}_2$$

for scalars k_1 and k_2. This shows that

$$\mathbf{v} = k_1\mathbf{e}_1 + k_2\mathbf{e}_2.$$

Taking dot product with $\mathbf{e}_1$ and $\mathbf{e}_2$, we get

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{e}_1 &= (k_1\mathbf{e}_1 + k_2\mathbf{e}_2)\cdot\mathbf{e}_1 \\
&= k_1(\mathbf{e}_1\cdot\mathbf{e}_1) + k_2(\mathbf{e}_2\cdot\mathbf{e}_1) \\
&= k_1\|\mathbf{e}_1\|^2 + k_2(0)
\end{aligned}$$

since $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal vectors, $\mathbf{e}_2\cdot\mathbf{e}_1 = 0$ and since $\mathbf{e}_1$ is a unit vector, $\|\mathbf{e}_1\| = 1$.

$$\Rightarrow \mathbf{v}\cdot\mathbf{e}_1 = k_1 \times 1 = k_1.$$

Similarly

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{e}_2 &= (k_1\mathbf{e}_1 + k_2\mathbf{e}_2)\cdot\mathbf{e}_2 \\
&= k_1(\mathbf{e}_1\cdot\mathbf{e}_2) + k_2(\mathbf{e}_2\cdot\mathbf{e}_2) \\
&= k_1(0) + k_2\|\mathbf{e}_2\|^2
\end{aligned}$$

since $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal vectors, $\mathbf{e}_1\cdot\mathbf{e}_2 = 0$ and since $\mathbf{e}_2$ is a unit vector, $\|\mathbf{e}_2\| = 1$.

$$\Rightarrow \mathbf{v}\cdot\mathbf{e}_2 = k_2 \times 1 = k_2.$$

Thus

$$\mathbf{v} = (\mathbf{v}\cdot\mathbf{e}_1)\mathbf{e}_1 + (\mathbf{v}\cdot\mathbf{e}_2)\mathbf{e}_2 \tag{14}$$

is the ***orthogonal decomposition*** of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$. The vectors

$$(\mathbf{v}\cdot\mathbf{e}_1)\mathbf{e}_1 \quad \text{and} \quad (\mathbf{v}\cdot\mathbf{e}_2)\mathbf{e}_2$$

are called the ***vector components*** $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$ respectively and

$$\mathbf{v}\cdot\mathbf{e}_1 \quad \text{and} \quad \mathbf{v}\cdot\mathbf{e}_2$$

are called the ***scalar components*** $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$ respectively.

Problem 2.8

Given that,

$$\mathbf{v} = \langle 2, 3 \rangle, \quad \mathbf{e}_1 = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle, \quad \text{and} \quad \mathbf{e}_2 = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle.$$

(a) Show that $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal unit vectors.

(b) Find the scalar components of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$.

(c) Find the vector components of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$.

(d) Verify equation (14).

Solution. (a) We have

$$\begin{aligned}
\mathbf{e}_1 \cdot \mathbf{e}_2 &= \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \cdot \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \\
&= \left(\frac{1}{\sqrt{2}}\right) \times \left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right) \times \left(\frac{1}{\sqrt{2}}\right) \\
&= -\frac{1}{2} + \frac{1}{2} \\
&= 0.
\end{aligned}$$

Thus $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal vectors. Since

$$\begin{aligned}
\|\mathbf{e}_1\| &= \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \\
&= \sqrt{\frac{1}{2} + \frac{1}{2}} \\
&= 1 \\
\|\mathbf{e}_2\| &= \sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \\
&= \sqrt{\frac{1}{2} + \frac{1}{2}} \\
&= 1
\end{aligned}$$

$\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors too.

(b) Scalar components of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$ are

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{e}_1 &= 2\left(\frac{1}{\sqrt{2}}\right) + 3\left(\frac{1}{\sqrt{2}}\right) = \frac{2+3}{\sqrt{2}} = \frac{5}{\sqrt{2}} \\
\mathbf{v} \cdot \mathbf{e}_2 &= 2\left(-\frac{1}{\sqrt{2}}\right) + 3\left(\frac{1}{\sqrt{2}}\right) = \frac{-2+3}{\sqrt{2}} = \frac{1}{\sqrt{2}}
\end{aligned}$$

(c) Vector components of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$ are

$$\begin{aligned}(\mathbf{v}\cdot\mathbf{e}_1)\,\mathbf{e}_1 &= \frac{5}{\sqrt{2}}\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle \\ &= \left\langle \frac{5}{2}, \frac{5}{2}\right\rangle \\ (\mathbf{v}\cdot\mathbf{e}_1)\,\mathbf{e}_2 &= \frac{1}{\sqrt{2}}\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle \\ &= \left\langle -\frac{1}{2}, \frac{1}{2}\right\rangle\end{aligned}$$

(d) Clearly

$$\begin{aligned}(\mathbf{v}\cdot\mathbf{e}_1)\,\mathbf{e}_1 + (\mathbf{v}\cdot\mathbf{e}_1)\,\mathbf{e}_2 &= \left\langle \frac{5}{2}, \frac{5}{2}\right\rangle + \left\langle -\frac{1}{2}, \frac{1}{2}\right\rangle \\ &= \left\langle \frac{5-1}{2}, \frac{5+1}{2}\right\rangle \\ &= \langle 2, 3\rangle \\ &= \mathbf{v}.\end{aligned}$$

■

2.2 ORTHOGONAL PROJECTIONS

The vector components of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$ in (14) are also called the ***orthogonal projections*** of $\mathbf{v}$ on $\mathbf{e}_1$ and $\mathbf{e}_2$ and are commonly denoted by

$$\text{proj}_{\mathbf{e}_1}\mathbf{v} = (\mathbf{v}\cdot\mathbf{e}_1)\mathbf{e}_1 \quad \text{and} \quad \text{proj}_{\mathbf{e}_2}\mathbf{v} = (\mathbf{v}\cdot\mathbf{e}_2)\mathbf{e}_2$$

In general if $\mathbf{e}$ is a unit vector, then the ***orthogonal projection of*** $\mathbf{v}$ ***on*** $\mathbf{e}$ is defined as

$$\text{proj}_{\mathbf{e}}\mathbf{v} = (\mathbf{v}\cdot\mathbf{e})\mathbf{e} \tag{15}$$

If $\mathbf{b}$ is a nonzero vector, then since $\dfrac{\mathbf{b}}{\|\mathbf{b}\|}$ is a unit vector in the direction of $\mathbf{b}$, the ***orthogonal projection of*** $\mathbf{v}$ ***on*** $\mathbf{b}$ is given by

$$\text{proj}_{\mathbf{b}}\mathbf{v} = \left(\mathbf{v}\cdot\frac{\mathbf{b}}{\|\mathbf{b}\|}\right)\left(\frac{\mathbf{b}}{\|\mathbf{b}\|}\right) = \left(\frac{\mathbf{v}\cdot\mathbf{b}}{\|\mathbf{b}\|}\right)\left(\frac{\mathbf{b}}{\|\mathbf{b}\|}\right) = \left(\frac{\mathbf{v}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\right)\mathbf{b}$$

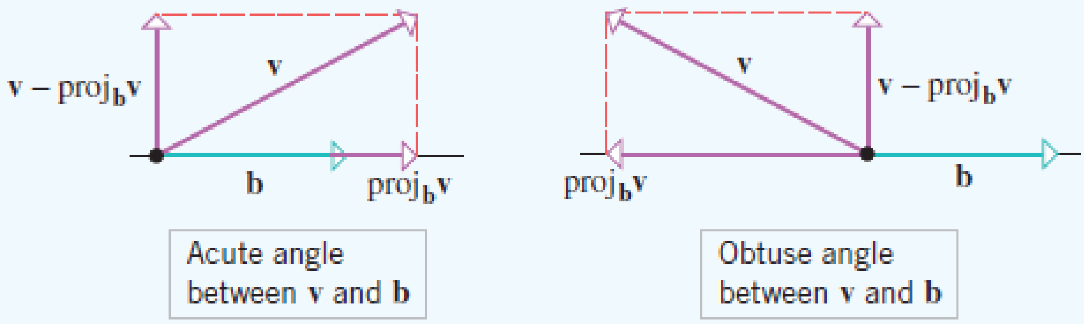
Thus we have

$$\boxed{\text{proj}_{\mathbf{b}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b} \qquad (16)}$$

▶ Geometrically, if $\mathbf{b}$ and $\mathbf{v}$ have a common initial point, then $\text{proj}_{\mathbf{b}}\mathbf{v}$ is the vector that is determined when a perpendicular is dropped from the terminal point of $\mathbf{v}$ to the line through $\mathbf{b}$.

▶ From Figure 2.12 it is clear that $\text{proj}_{\mathbf{b}}\mathbf{v}$ is a vector parallel to $\mathbf{b}$.

Figure 2.12



Moreover, it is evident from Figure 2.12 that if we subtract $\text{proj}_{\mathbf{b}}\mathbf{v}$ from $\mathbf{v}$, then the resulting vector

$$\mathbf{v} - \text{proj}_{\mathbf{b}}\mathbf{v}$$

will be orthogonal to $\mathbf{b}$; we call this the ***vector component of*** $\mathbf{v}$ ***orthogonal to*** $\mathbf{b}$. As a summary we have

Suppose a nonzero vector $\mathbf{b}$ is given. Then for any vector $\mathbf{v}$,

- ▶ The orthogonal projection of $\mathbf{v}$ on $\mathbf{b}$ is

$$\text{proj}_{\mathbf{b}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b}$$

- ▶ The vector component of $\mathbf{v}$ orthogonal to $\mathbf{b}$ is given by

$$\mathbf{v} - \text{proj}_{\mathbf{b}}\mathbf{v}$$

- ▶ If $\mathbf{w}_1 = \text{proj}_{\mathbf{b}}\mathbf{v}$ and $\mathbf{w}_2 = \mathbf{v} - \text{proj}_{\mathbf{b}}\mathbf{v}$ then $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal vectors and

$$\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2.$$

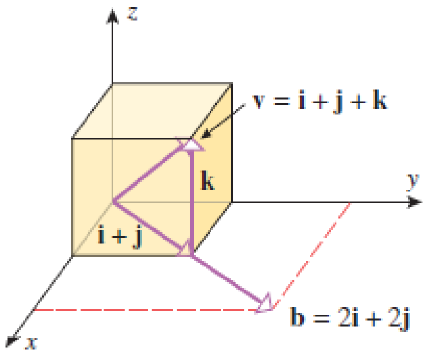
  Moreover, $\mathbf{w}_1$ is parallel to $\mathbf{b}$ and $\mathbf{w}_2$ is perpendicular to $\mathbf{b}$.

Problem 2.9

Find the orthogonal projection of $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ on $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j}$, and then find the vector component of $\mathbf{v}$ orthogonal to $\mathbf{b}$.

Solution. We have

$$
\begin{aligned}
\mathbf{v}\cdot\mathbf{b}(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(2\mathbf{i}+2\mathbf{j})&\\
&=1\times 2+1\times 2+1\times 0\\
&=2+2+0\\
&=4\\
\|\mathbf{b}\|^2&=2^2+2^2\\
&=8.
\end{aligned}
$$



Thus, the orthogonal projection of $\mathbf{v}$ on $\mathbf{b}$ is

$$
\begin{aligned}
\operatorname{proj}_{\mathbf{b}}\mathbf{v}&=\frac{\mathbf{v}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b}\\
&=\frac{4}{8}(2\mathbf{i}+2\mathbf{j})\\
&=\frac{1}{2}(2\mathbf{i}+2\mathbf{j})\\
&=\mathbf{i}+\mathbf{j}
\end{aligned}
$$

and the vector component of $\mathbf{v}$ orthogonal to $\mathbf{b}$ is

$$\mathbf{v}-\operatorname{proj}_{\mathbf{b}}\mathbf{v}=(\mathbf{i}+\mathbf{j}+\mathbf{k})-(\mathbf{i}+\mathbf{j})=\mathbf{k}.$$

■

Problem 2.10

In each part, find the vector component of $\mathbf{v}$ along $\mathbf{b}$ and the vector component of $\mathbf{v}$ orthogonal to $\mathbf{b}$. Then sketch the vectors $\mathbf{v}, \operatorname{proj}_{\mathbf{b}}\mathbf{v}$, and $\mathbf{v}-\operatorname{proj}_{\mathbf{b}}\mathbf{v}$.

(a) $\mathbf{v}=2\mathbf{i}-\mathbf{j},\ \mathbf{b}=3\mathbf{i}+4\mathbf{j}$

(b) $\mathbf{v}=\langle 4,5\rangle,\ \mathbf{b}=\langle 1,-2\rangle$

(c) $\mathbf{v}=-3\mathbf{i}-2\mathbf{j},\ \mathbf{b}=2\mathbf{i}+\mathbf{j}$

Solution. In each part we find $\operatorname{proj}_{\mathbf{b}}\mathbf{v}$ and $\mathbf{v}-\operatorname{proj}_{\mathbf{b}}\mathbf{v}$.
(a) We have

$$
\begin{aligned}
\mathbf{v}\cdot\mathbf{b}&=(2\mathbf{i}-\mathbf{j})\cdot(3\mathbf{i}+4\mathbf{j})\\
&=2\times 3+(-1)\times 4\\
&=6-4\\
&=2\\
\|\mathbf{b}\|^2&=3^2+4^2\\
&=25.
\end{aligned}
$$

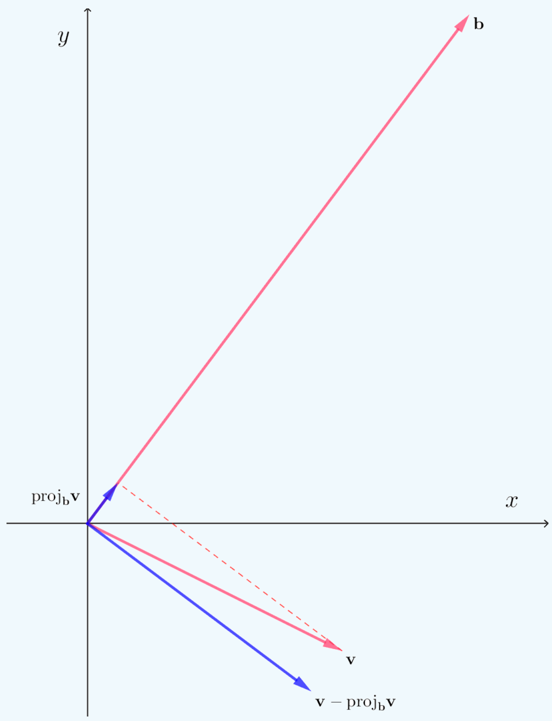
Thus, vector component of of $\mathbf{v}$ along $\mathbf{b}$ is

$$\begin{aligned}
\operatorname{proj}_{\mathbf{b}}\mathbf{v} &= \frac{\mathbf{v}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b} \\
&= \frac{2}{25}(3\mathbf{i}+4\mathbf{j}) \\
&= \frac{6}{25}\mathbf{i}+\frac{8}{25}\mathbf{j}
\end{aligned}$$

and the vector component of $\mathbf{v}$ orthogonal to $\mathbf{b}$ is

$$\begin{aligned}
\mathbf{v}-\operatorname{proj}_{\mathbf{b}}\mathbf{v} &= (2\mathbf{i}-\mathbf{j}) - \left(\frac{6}{25}\mathbf{i}+\frac{8}{25}\mathbf{j}\right) \\
&= \left(2-\frac{6}{25}\right)\mathbf{i} + \left(-1-\frac{8}{25}\right)\mathbf{j} \\
&= \frac{44}{25}\mathbf{i}-\frac{33}{25}\mathbf{j}.
\end{aligned}$$

Figure 2.13



(b) We have

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{b} &= \langle 4,5\rangle\cdot\langle 1,-2\rangle \\
&= 4\times 1+5\times(-2) \\
&= 4-10 \\
&= -6 \\
\|\mathbf{b}\|^2 &= 1^2+(-2)^2 \\
&= 5.
\end{aligned}$$

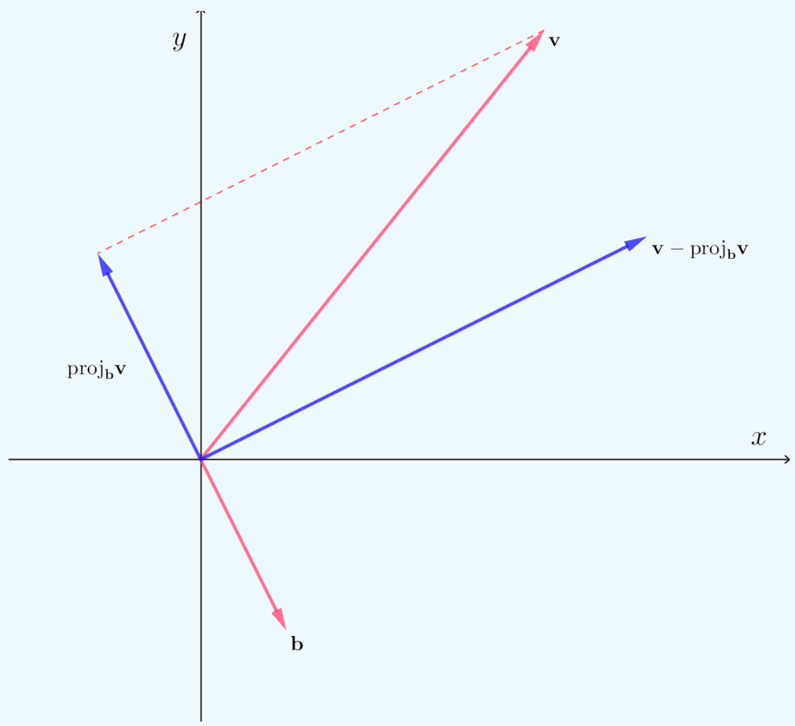
Thus, vector component of of $\mathbf{v}$ along $\mathbf{b}$ is

$$\begin{aligned}\text{proj}_{\mathbf{b}}\mathbf{v} &= \frac{\mathbf{v}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b}\\ &= -\frac{6}{5}\langle 1,-2\rangle\\ &= \left\langle -\frac{6}{5},\frac{12}{5}\right\rangle\end{aligned}$$

and the vector component of $\mathbf{v}$ orthogonal to $\mathbf{b}$ is

$$\begin{aligned}\mathbf{v}-\text{proj}_{\mathbf{b}}\mathbf{v} &= \langle 4,5\rangle - \left\langle -\frac{6}{5},\frac{12}{5}\right\rangle\\ &= \left\langle 4+\frac{6}{5}, 5-\frac{12}{5}\right\rangle\\ &= \left\langle \frac{26}{5},\frac{13}{5}\right\rangle.\end{aligned}$$

Figure 2.14



(c) We have

$$\begin{aligned}\mathbf{v}\cdot\mathbf{b} &= (-3\mathbf{i}-2\mathbf{j})\cdot(2\mathbf{i}+\mathbf{j})\\ &= (-3)\times 2 + (-2)\times 1\\ &= -6-2\\ &= -8\\ \|\mathbf{b}\|^2 &= 2^2+1^2\\ &= 5.\end{aligned}$$

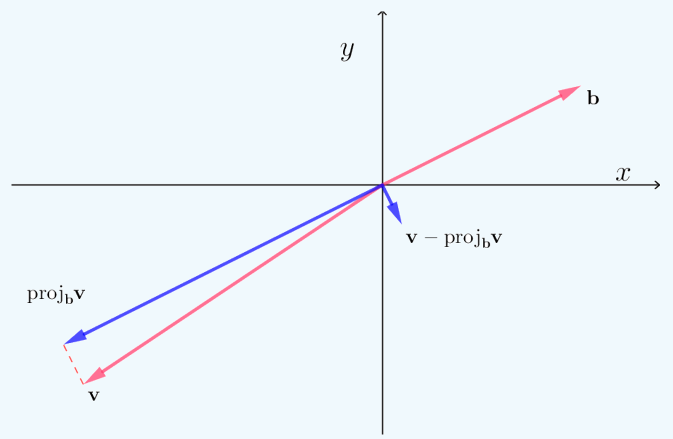
Thus, vector component of of $\mathbf{v}$ along $\mathbf{b}$ is

$$\begin{aligned}\operatorname{proj}_{\mathbf{b}}\mathbf{v} &= \frac{\mathbf{v}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b}\\ &= \frac{-8}{5}(2\mathbf{i}+\mathbf{j})\\ &= -\frac{16}{5}\mathbf{i}-\frac{8}{5}\mathbf{j}\end{aligned}$$

and the vector component of $\mathbf{v}$ orthogonal to $\mathbf{b}$ is

$$\begin{aligned}\mathbf{v}-\operatorname{proj}_{\mathbf{b}}\mathbf{v} &= (-3\mathbf{i}-2\mathbf{j})-\left(-\frac{16}{5}\mathbf{i}-\frac{8}{5}\mathbf{j}\right)\\ &= \left(-3+\frac{16}{5}\right)\mathbf{i}+\left(-2+\frac{8}{5}\right)\mathbf{j}\\ &= \frac{1}{5}\mathbf{i}-\frac{2}{5}\mathbf{j}\\ &= \frac{1}{5}(\mathbf{i}-2\mathbf{j}).\end{aligned}$$

Figure 2.15



Problem 2.11

In each part, find the vector component of $\mathbf{v}$ along $\mathbf{b}$ and the vector component of $\mathbf{v}$ orthogonal to $\mathbf{b}$.

(a) $\mathbf{v}=2\mathbf{i}-\mathbf{j}+3\mathbf{k},\ \mathbf{b}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}$

(b) $\mathbf{v}=\langle 4,-1,7\rangle,\ \mathbf{b}=\langle 2,3,-6\rangle$

Solution. (a) We have

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{b} &= (2\mathbf{i}-\mathbf{j}+3\mathbf{k})\cdot(\mathbf{i}+2\mathbf{j}+2\mathbf{k})\\
&= 2\times 1+(-1)\times 2+3\times 2\\
&= 2-2+6\\
&= 6\\
\|\mathbf{b}\|^2 &= 1^2+2^2+2^2\\
&= 9.
\end{aligned}$$

Thus, vector component of of $\mathbf{v}$ along $\mathbf{b}$ is

$$\begin{aligned}
\text{proj}_{\mathbf{b}}\mathbf{v} &= \frac{\mathbf{v}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b}\\
&= \frac{6}{9}(\mathbf{i}+2\mathbf{j}+2\mathbf{k})\\
&= \frac{2}{3}\mathbf{i}+\frac{4}{3}\mathbf{j}+\frac{4}{3}\mathbf{k}
\end{aligned}$$

and the vector component of $\mathbf{v}$ orthogonal to $\mathbf{b}$ is

$$\begin{aligned}
\mathbf{v}-\text{proj}_{\mathbf{b}}\mathbf{v} &= (2\mathbf{i}-\mathbf{j}+3\mathbf{k})-\left(\frac{2}{3}\mathbf{i}+\frac{4}{3}\mathbf{j}+\frac{4}{3}\mathbf{k}\right)\\
&= \left(2-\frac{2}{3}\right)\mathbf{i}+\left(-1-\frac{4}{3}\right)\mathbf{j}+\left(3-\frac{4}{3}\right)\mathbf{k}\\
&= \frac{4}{3}\mathbf{i}-\frac{7}{3}\mathbf{j}+\frac{5}{3}\mathbf{k}.
\end{aligned}$$

(b) We have

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{b} &= \langle 4,-1,7\rangle\cdot\langle 2,3,-6\rangle\\
&= 4\times 2+(-1)\times 3+7\times(-6)\\
&= 8-3-42\\
&= -37\\
\|\mathbf{b}\|^2 &= 2^2+3^2+(-6)^2\\
&= 49.
\end{aligned}$$

Thus, vector component of of $\mathbf{v}$ along $\mathbf{b}$ is

$$\begin{aligned}
\text{proj}_{\mathbf{b}}\mathbf{v} &= \frac{\mathbf{v}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b}\\
&= -\frac{37}{49}\langle 2,3,-6\rangle\\
&= \frac{37}{49}\langle -2,-3,6\rangle
\end{aligned}$$

and the vector component of $\mathbf{v}$ orthogonal to $\mathbf{b}$ is

$$\begin{aligned}\mathbf{v}-\mathrm{proj}_{\mathbf{b}}\mathbf{v} &= \langle 4,-1,7\rangle - \frac{37}{49}\langle -2,-3,6\rangle \\ &= \left\langle 4+\frac{74}{49}, -1+\frac{111}{49}, 7-\frac{222}{49}\right\rangle \\ &= \frac{1}{49}\langle 270,62,121\rangle .\end{aligned}$$

Problem 2.12

Express the vector $\mathbf{v}$ as the sum of a vector parallel to $\mathbf{b}$ and a vector orthogonal to $\mathbf{b}$.

(a) $\mathbf{v}=2\mathbf{i}-4\mathbf{j},\ \mathbf{b}=\mathbf{i}+\mathbf{j}$

(b) $\mathbf{v}=3\mathbf{i}+\mathbf{j}-2\mathbf{k},\ \mathbf{b}=2\mathbf{i}-\mathbf{k}$

(c) $\mathbf{v}=4\mathbf{i}-2\mathbf{j}+6\mathbf{k},\ \mathbf{b}=-2\mathbf{i}+\mathbf{j}-3\mathbf{k}$

(d) $\mathbf{v}=\langle -3,5\rangle,\ \mathbf{b}=\langle 1,1\rangle$

(e) $\mathbf{v}=\langle -2,1,6\rangle,\ \mathbf{b}=\langle 0,-2,1\rangle$

(f) $\mathbf{v}=\langle 1,3,1\rangle,\ \mathbf{b}=\langle 3,-2,5\rangle$

Solution. In each case we find $\mathbf{w}_1=\mathrm{proj}_{\mathbf{b}}\mathbf{v}$ and $\mathbf{w}_2=\mathbf{v}-\mathrm{proj}_{\mathbf{b}}\mathbf{v}$. Then $\mathbf{w}_1$ is parallel to $\mathbf{b}$, $\mathbf{w}_2$ is orthogonal to $\mathbf{b}$ and

$$\mathbf{v}=\mathbf{w}_1+\mathbf{w}_2.$$

(a) We have

$$\begin{aligned}\mathbf{v}\cdot\mathbf{b} &= (2\mathbf{i}-4\mathbf{j})\cdot(\mathbf{i}+\mathbf{j}) \\ &= 2\times 1+(-4)\times 1 \\ &= 2-4 \\ &= -2 \\ \|\mathbf{b}\|^2 &= 1^2+1^2 \\ &= 2.\end{aligned}$$

Thus, vector component of of $\mathbf{v}$ along $\mathbf{b}$ is

$$\begin{aligned}\mathrm{proj}_{\mathbf{b}}\mathbf{v} &= \frac{\mathbf{v}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b} \\ &= -\frac{2}{2}(\mathbf{i}+\mathbf{j}) \\ &= -\mathbf{i}-\mathbf{j}\end{aligned}$$

and the vector component of $\mathbf{v}$ orthogonal to $\mathbf{b}$ is

$$\begin{aligned}\mathbf{v} - \text{proj}_{\mathbf{b}}\mathbf{v} &= (2\mathbf{i} - 4\mathbf{j}) - (-\mathbf{i} - \mathbf{j}) \\ &= 3\mathbf{i} - 3\mathbf{j}.\end{aligned}$$

Thus

$$\mathbf{w}_1 = -\mathbf{i} - \mathbf{j} \quad \text{and} \quad \mathbf{w}_2 = 3\mathbf{i} - 3\mathbf{j}.$$

It is clear that

$$\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} = (-\mathbf{i} - \mathbf{j}) + (3\mathbf{i} - 3\mathbf{j}) = \mathbf{w}_1 + \mathbf{w}_2.$$

(b) We have

$$\begin{aligned}\mathbf{v} \cdot \mathbf{b} &= (3\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \cdot (2\mathbf{i} - \mathbf{k}) \\ &= 3 \times 2 + 1 \times 0 + (-2) \times (-1) \\ &= 6 + 0 + 2 \\ &= 8 \\ \|\mathbf{b}\|^2 &= 2^2 + 0^2 + (-1)^2 \\ &= 5.\end{aligned}$$

Thus, vector component of of $\mathbf{v}$ along $\mathbf{b}$ is

$$\begin{aligned}\text{proj}_{\mathbf{b}}\mathbf{v} &= \frac{\mathbf{v} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b} \\ &= \frac{8}{5}(2\mathbf{i} - \mathbf{k}) \\ &= \frac{16}{5}\mathbf{i} - \frac{8}{5}\mathbf{k}\end{aligned}$$

and the vector component of $\mathbf{v}$ orthogonal to $\mathbf{b}$ is

$$\begin{aligned}\mathbf{v} - \text{proj}_{\mathbf{b}}\mathbf{v} &= (3\mathbf{i} + \mathbf{j} - 2\mathbf{k}) - \left(\frac{16}{5}\mathbf{i} - \frac{8}{5}\mathbf{k}\right) \\ &= \left(3 - \frac{16}{5}\right)\mathbf{i} + (1 - 0)\mathbf{j} + \left(-2 + \frac{8}{5}\right)\mathbf{k} \\ &= -\frac{1}{5}\mathbf{i} + \mathbf{j} - \frac{2}{5}\mathbf{k}\end{aligned}$$

Thus

$$\mathbf{w}_1 = \frac{16}{5}\mathbf{i} - \frac{8}{5}\mathbf{k} \quad \text{and} \quad \mathbf{w}_2 = -\frac{1}{5}\mathbf{i} + \mathbf{j} - \frac{2}{5}\mathbf{k}.$$

It is clear that

$$\mathbf{v} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k} = \left(\frac{16}{5}\mathbf{i} - \frac{8}{5}\mathbf{k}\right) + \left(-\frac{1}{5}\mathbf{i} + \mathbf{j} - \frac{2}{5}\mathbf{k}\right) = \mathbf{w}_1 + \mathbf{w}_2.$$

The problems (c) to (e) are left as exercise.

(f) We have

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{b} &= \langle 1,3,1\rangle\cdot\langle 3,-2,5\rangle\\
&= 1\times 3+3\times(-2)+1\times 5\\
&= 3-6+5\\
&= 2\\
\|\mathbf{b}\|^2 &= 3^2+(-2)^2+5^2\\
&= 38.
\end{aligned}$$

Thus, vector component of of $\mathbf{v}$ along $\mathbf{b}$ is

$$\begin{aligned}
\mathrm{proj}_{\mathbf{b}}\mathbf{v} &= \frac{\mathbf{v}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b}\\
&= \frac{2}{38}\langle 3,-2,5\rangle\\
&= \frac{1}{19}\langle 3,-2,5\rangle\\
&= \left\langle \frac{3}{19},\frac{-2}{19},\frac{5}{19}\right\rangle
\end{aligned}$$

and the vector component of $\mathbf{v}$ orthogonal to $\mathbf{b}$ is

$$\begin{aligned}
\mathbf{v}-\mathrm{proj}_{\mathbf{b}}\mathbf{v} &= \langle 1,3,1\rangle-\left\langle \frac{3}{19},\frac{-2}{19},\frac{5}{19}\right\rangle\\
&= \left\langle 1-\frac{3}{19},3+\frac{2}{19},1-\frac{5}{19}\right\rangle\\
&= \left\langle \frac{16}{19},\frac{59}{19},\frac{14}{19}\right\rangle.
\end{aligned}$$

Thus

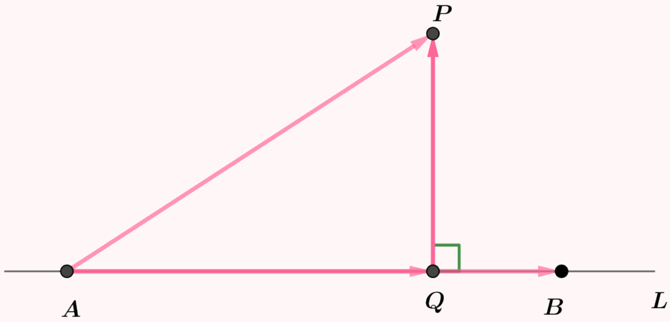
$$\mathbf{w}_1=\left\langle \frac{3}{19},\frac{-2}{19},\frac{5}{19}\right\rangle \quad \text{and} \quad \mathbf{w}_2=\left\langle \frac{16}{19},\frac{59}{19},\frac{14}{19}\right\rangle.$$

It is clear that

$$\mathbf{v}=\langle 1,3,1\rangle=\left\langle \frac{3}{19},\frac{-2}{19},\frac{5}{19}\right\rangle+\left\langle \frac{16}{19},\frac{59}{19},\frac{14}{19}\right\rangle=\mathbf{w}_1+\mathbf{w}_2. \qquad \blacksquare$$

Problem 2.13

If L is a line in 2-space or 3-space that passes through the points A and B, then the distance from a point P to the line L is equal to the length of the component of the vector $\overrightarrow{AP}$ that is orthogonal to the vector $\overrightarrow{AB}$.



Solution. From the figure,

$$\overrightarrow{AQ} = \text{proj}_{\overrightarrow{AB}}\overrightarrow{AP} = \text{proj}_{\overrightarrow{AB}}\overrightarrow{AP} = \frac{\overrightarrow{AP} \cdot \overrightarrow{AB}}{\left\|\overrightarrow{AB}\right\|^2}\overrightarrow{AB}$$

and

$$\overrightarrow{PQ} = \overrightarrow{AP} - \overrightarrow{AQ}$$

That is $\overrightarrow{AQ}$ is component of the vector $\overrightarrow{AP}$ that is orthogonal to the vector $\overrightarrow{AB}$. Moreover it is clear that distance of P to L is,

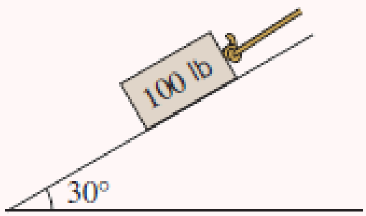
$$d = \|PQ\|.$$

Hence

$$d = \left\|\overrightarrow{AP} - \frac{\overrightarrow{AP} \cdot \overrightarrow{AB}}{\left\|\overrightarrow{AB}\right\|^2}\overrightarrow{AB}\right\|. \tag{17}$$

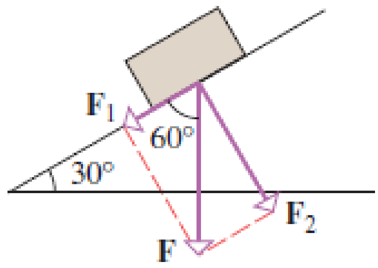
Problem 2.14

A rope is attached to a $100\,lb$ block on a ramp that is inclined at an angle of $30°$ with the ground. How much force does the block exert against the ramp, and how much force must be applied to the rope in a direction parallel to the ramp to prevent the block from sliding down the ramp? (Assume that the ramp is smooth, that is, exerts no frictional forces.)



Solution. Let $\mathbf{F}$ denote the downward force of gravity on the block (so $\|\mathbf{F}\| = 100\,lb$), and let $\mathbf{F}_1$ and $\mathbf{F}_2$ be the vector components of $\mathbf{F}$ parallel and perpendicular to the ramp. The lengths of $\mathbf{F}_1$ and $\mathbf{F}_2$ are

$$\begin{aligned}
\|\mathbf{F}_1\| &= \|\mathbf{F}\| \cos 60^\circ \\
&= 100\left(\frac{1}{2}\right) \\
&= 50\,lb \\
\|\mathbf{F}_1\| &= \|\mathbf{F}\| \sin 60^\circ \\
&= 100\left(\frac{\sqrt{3}}{2}\right) \\
&\approx 86.6\,lb.
\end{aligned}$$



Thus, the block exerts a force of approximately $86.6\,lb$ against the ramp, and it requires a force of $50\,lb$ to prevent the block from sliding down the ramp. ■

WORK

If a force vector $\mathbf{F}$ of constant magnitude is applied on an object moving along a line from the point P to a point Q where $\overrightarrow{PQ}$ makes an angle θ with $\mathbf{F}$, then work W done by the force on the object is given by

$$W = \|\mathbf{F}\| \left\|\overrightarrow{PQ}\right\| \cos\theta = \mathbf{F} \cdot \overrightarrow{PQ} \tag{18}$$

If the force vector is applied in the direction of $\overrightarrow{PQ}$, then $\theta = 0$ so that work done,

$$W = \|\mathbf{F}\| \left\|\overrightarrow{PQ}\right\|$$

Problem 2.15

(a) A wagon is pulled horizontally by exerting a constant force of $10\,lb$ on the handle at an angle of 60° with the horizontal. How much work is done in moving the wagon $50\,ft$?

(b) Aforce of $\mathbf{F} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}\,lb$ is applied to a point that moves on a line from $P(-1, 1, 2)$ to $Q(3, 0, -2)$. If distance is measured in feet, how much work is done?

Solution. (a) Here $\|\mathbf{F}\| = 10\, lb, \theta = 60^\circ$ and $\left\|\overrightarrow{PQ}\right\| = 50\, ft$. Then the work done is

$$\begin{aligned} W &= \|\mathbf{F}\| \left\|\overrightarrow{PQ}\right\| \cos\theta \\ &= 10 \times 50 \times \cos 60^\circ \\ &= 500 \times \frac{1}{2} \\ &= 250\, ft\, lb. \end{aligned}$$

(b) We have

$$\overrightarrow{PQ} = (3-(-1))\mathbf{i} + (0-1)\mathbf{j} + (-2-2)\mathbf{k} = 4\mathbf{i} - \mathbf{j} - 4\mathbf{k}.$$

Hence the work done is

$$\begin{aligned} W &= \mathbf{F} \cdot \overrightarrow{PQ} \\ &= (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (4\mathbf{i} - \mathbf{j} - 4\mathbf{k})) \\ &= 3 \times 4 + (-1) \times (-1) + 2 \times (-4) \\ &= 12 + 1 - 8 \\ &= 5\, ft\, lb. \end{aligned}$$

■

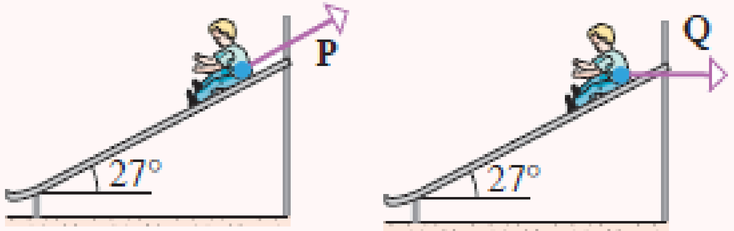
Problem 2.16

As shown in the accompanying figure, a child with mass $34\, kg$ is seated on a smooth (frictionless) playground slide that is inclined at an angle of 27° with the horizontal.

(a) Estimate the force that the child exerts on the slide, and estimate how much force must be applied in the direction of P to prevent the child from sliding down the slide.

(b) Estimate how much force must be applied in the direction of Q to prevent the child from sliding down the slide?

Take the acceleration due to gravity to be $9.8\, m/s^2$.

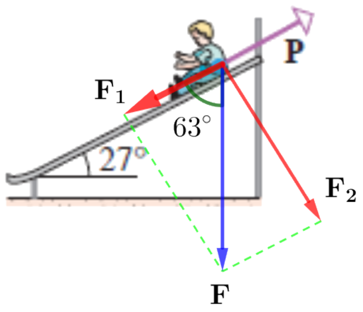


Solution. Let $\mathbf{F}$ denote the downward force of gravity on the child. Then

$$\|\mathbf{F}\| = 34 \times 9.8 = 333.2\, N.$$

(a) Suppose $\mathbf{F}_1$ and $\mathbf{F}_2$ be the vector components of $\mathbf{F}$ parallel and perpendicular to the slide. Then

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2.$$

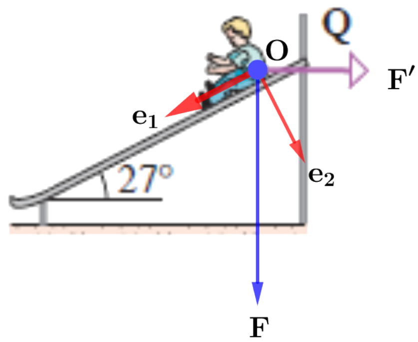


From the figure

$$\|\mathbf{F}_1\| = \|\mathbf{F}\| \cos 63^\circ \approx 333.2 \times 0.454 = 151.27\,N$$
$$\|\mathbf{F}_2\| = \|\mathbf{F}\| \sin 63^\circ \approx 333.2 \times 0.891 = 296.88\,N$$

Hence amount of force to be applied in the direction of P to prevent the child from sliding down the slide is $151.27\,N$.

(b) We proceed as follows. Choose O as the origin, direction of Q as positive x-direction and introduce a rectangular coordinate system. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be unit vectors parallel and perpendicular to the slide. Then $\mathbf{e}_1$ makes an angle $(180+27)^\circ$ and $\mathbf{e}_2$ makes an angle $(270+27)^\circ$ with the positive x-axis.



Thus

$$\mathbf{e}_1 = \langle \cos(180+27)^\circ, \sin(180+27)^\circ \rangle = \langle -\cos 27^\circ, -\sin 27^\circ \rangle$$

and

$$\mathbf{e}_2 = \langle \cos(270+27)^\circ, \sin(270+27)^\circ \rangle = \langle \sin 27^\circ, -\cos 27^\circ \rangle .$$

Let $\mathbf{F'}$ be the force to be applied in the direction of Q to prevent the child from sliding down the slide and suppose $\|\mathbf{F'}\| = a$. Then the forces applied on the child are, $333.2\,N$ force due to gravity, towards negative y- direction and $\mathbf{F'}$ of magnitude a towards positive x-direction. So net force applied is

$$\mathbf{F''} = a\mathbf{i} - 333.2\mathbf{j} = \langle a, -333.2 \rangle .$$

The net effective force exerted in the direction parallel to the slide is same as the component of $\mathbf{F''}$ in the direction of $\mathbf{e}_1$ which has magnitude

$$\mathbf{F''} \cdot \mathbf{e}_1 = \langle a, -333.2 \rangle \cdot \langle -\cos 27^\circ, -\sin 27^\circ \rangle = -a\cos 27^\circ + 333.2 \sin 27^\circ.$$

In order to prevent the child from sliding down the child, $\mathbf{F''} \cdot \mathbf{e}_1$ must be non-positive. (if it is positive, since the direction of this force is downward along the slide, the child will slide down.) That is

$$\begin{aligned} -a\cos 27^\circ + 333.2 \sin 27^\circ &\leq 0 \\ \Rightarrow a\cos 27^\circ &\geq 333.2 \sin 27^\circ \\ \Rightarrow a &\geq 333.2 \tan 27^\circ \\ &\approx 333.2 \times 0.5095 \\ &= 169.77\,N. \end{aligned}$$

Hence minimum amount of force to be applied in the direction of Q to prevent the child from sliding down the slide is $169.77\,N$. ■

Problem 2.17

Suppose that the slide in Problem 2.16 is $4\,m$ long. Estimate the work done by gravity if the child slides from the top of the slide to the bottom.

Solution. Here the force is applied in the direction of $\mathbf{F}_1$ and since the length of slide is $4\,m$, the work done by gravity if the child slides from the top of the slide to the bottom is given by

$$W = \|\mathbf{F}_1\| \times 4 = 151.27 \times 4 = 605.08\,Nm = 605.08\,\mathrm{J}.$$ ■

Problem 2.18

A boat travels 100 meters due north while the wind exerts a force of 500 newtons toward the northeast. How much work does the wind do?

Solution. Here

$$\|\mathbf{F}\| = 500, \quad \left\|\overrightarrow{PQ}\right\| = 100.$$

Also angle between the direction of motion of boat and the wind is $45°$. (since the angle between north and northeast is $45°$.). Hence work done is

$$\begin{aligned} W &= \|\mathbf{F}\| \left\|\overrightarrow{PQ}\right\| \cos\theta \\ &= 500 \times 100 \times \cos 45° \\ &= 50000 \times \frac{1}{\sqrt{2}} \\ &= 50000 \times \frac{\sqrt{2}}{2} \\ &= 25000\sqrt{2} \\ &\approx 35355.34\mathrm{J}. \end{aligned}$$ ■

Problem 2.19

Prove that

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\,\|\mathbf{u}\|^2 + 2\,\|\mathbf{v}\|^2 \tag{19}$$

and interpret the result geometrically by translating it into a theorem about parallelograms.

Solution. From (9)

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} \quad \Rightarrow \|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}.$$

Using this we have

$$
\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \\
&= \mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v} \\
&= \|\mathbf{u}\|^2+\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2 \\
&= \|\mathbf{u}\|^2+2(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|^2
\end{aligned}
$$

and

$$
\begin{aligned}
\|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\
&= \mathbf{u}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot(-\mathbf{v}) \\
&= \|\mathbf{u}\|^2-\mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v} \\
&= \|\mathbf{u}\|^2-2(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|^2
\end{aligned}
$$

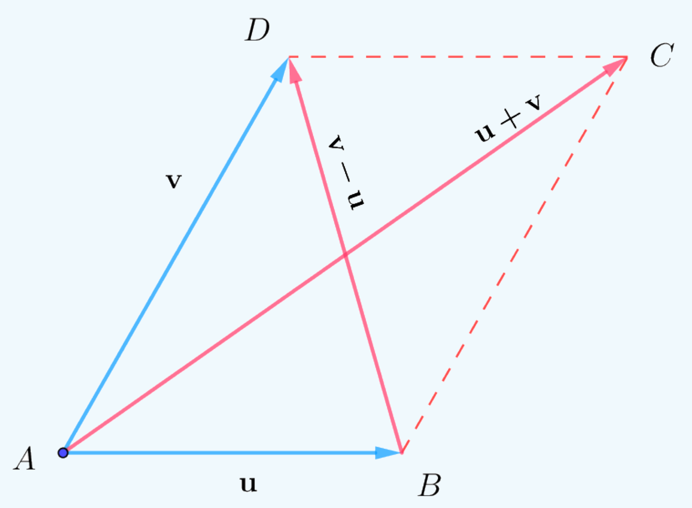
Hence

$$
\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2+\|\mathbf{u}-\mathbf{v}\|^2 &= \left(\|\mathbf{u}\|^2+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2\right)+\left(\|\mathbf{u}\|^2-2(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|^2\right) \\
&= 2\,\|\mathbf{u}\|^2+2\,\|\mathbf{v}\|^2 .
\end{aligned}
$$

Consider $\mathbf{u}$ and $\mathbf{v}$ as vectors with same initial point. Then the vectors $\mathbf{u}+\mathbf{v}$ and $\mathbf{v}-\mathbf{u}$ form the diagonals of a parallelogram $ABCD$ whose adjacent sides are along $\mathbf{u}$ and $\mathbf{v}$ (see the following figure). Then it is clear that

$$
\begin{aligned}
CD = AB &= \|\mathbf{u}\| & \qquad BC = AD &= \|\mathbf{v}\| \\
AC &= \|\mathbf{u}+\mathbf{v}\| & \qquad BD &= \|\mathbf{v}-\mathbf{u}\| = \|\mathbf{u}-\mathbf{v}\|
\end{aligned}
$$

Figure 2.16



For the parallelogram $ABCD$, we know that

$$\begin{aligned} AC^2 + BD^2 &= AB^2 + BC^2 + CD^2 + AD^2 \\ &= 2AB^2 + 2AD^2 \\ \Rightarrow \|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 &= 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2 . \end{aligned}$$

Usually the relation between the norms of vectors $\mathbf{u}, \mathbf{v}, \mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ as proved above is called the ***parallelogram law***. ■

Problem 2.20

Prove that

$$\mathbf{u}\cdot\mathbf{v} = \frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2 .$$

Solution. From the solution of Problem 2.19, we have

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2$$

and

$$\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 - 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2$$

Subtracting we get

$$\begin{aligned} \|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 &= 2(\mathbf{u}\cdot\mathbf{v}) - (-2(\mathbf{u}\cdot\mathbf{v})) \\ &= 4(\mathbf{u}\cdot\mathbf{v}) \end{aligned}$$

which gives the required relation. ■

2.3 CROSS PRODUCT

If $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ are vectors in 3-space, then the cross product $\mathbf{u}\times\mathbf{v}$ is the vector defined by

$$\mathbf{u}\times\mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} \tag{20}$$

or, equivalently

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{21}$$

▶ The cross product of two vectors is a vector.

The following are some properties of cross product.

Theorem 2.6

If $\mathbf{u}, \mathbf{v}$, *and* $\mathbf{w}$ *are vectors in 3-space and* k *is a scalar, then:*

(a) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(b) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

(c) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(d) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

(e) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(f) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

(g) $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$

(h) $\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{i} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$

(i) $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$

Proof. Suppose, $\mathbf{u} = \langle u_1, u_2, u_3 \rangle, \mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$. Then,

(a) Using the properties of determinants, we have

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= -\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} \\ &= -(\mathbf{v} \times \mathbf{u}).\end{aligned}$$

(b) Using the properties of determinants, we have

$$\begin{aligned}\mathbf{u} \times (\mathbf{v} + \mathbf{w}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 + w_1 & v_2 + w_2 & v_3 + w_3 \end{vmatrix} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})\end{aligned}$$

(c) Proof similar to (b), left as an exercise.

(d) If k is a scalar then using property of determinants, we have

$$\begin{aligned} k(\mathbf{u} \times \mathbf{v}) &= k \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ ku_1 & ku_2 & ku_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= (k\mathbf{u}) \times \mathbf{v} \end{aligned}$$

Similarly,

$$\begin{aligned} k(\mathbf{u} \times \mathbf{v}) &= k \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ kv_1 & kv_2 & kv_3 \end{vmatrix} \\ &= \mathbf{u} \times (k\mathbf{v}) \end{aligned}$$

(e) Since $\mathbf{0} = \langle 0, 0, 0 \rangle$, we have

$$\begin{aligned} \mathbf{u} \times \mathbf{0} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ 0 & 0 & 0 \end{vmatrix} \\ &= \mathbf{0} \end{aligned}$$

(f) Again using property of determinants, we have

$$\begin{aligned} \mathbf{u} \times \mathbf{u} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix} \\ &= \mathbf{0}. \end{aligned}$$

(g) We have

$$\mathbf{i} = \langle 1, 0, 0 \rangle\,, \quad \mathbf{j} = \langle 0, 1, 0 \rangle\,, \quad \mathbf{k} = \langle 0, 0, 1 \rangle$$

so that

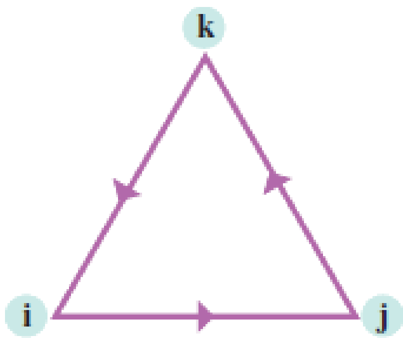
$$\mathbf{i}\times\mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$
$$= -1(0\times\mathbf{j} - 1\times\mathbf{k}) \quad \text{(expanding along second row)}$$
$$= \mathbf{k}.$$

Others left as exercise.

(h) Proof left as exercise.

(i) Use (f). ■

▶ In this diagram, the cross product of two consecutive vectors in the counterclockwise direction is the next vector around, and the cross product of two consecutive vectors in the clockwise direction is the negative of the next vector around.



GEOMETRIC PROPERTIES OF THE CROSS PRODUCT

The following theorem shows that the cross product of two vectors is orthogonal to both factors.

Theorem 2.7

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in 3-space, then:

(a) $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = 0$ ($\mathbf{u}\times\mathbf{v}$ *is orthogonal to* $\mathbf{u}$)

(b) $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v}) = 0$ ($\mathbf{u}\times\mathbf{v}$ *is orthogonal to* $\mathbf{v}$)

Proof. We assume that $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. Then by (20), we have

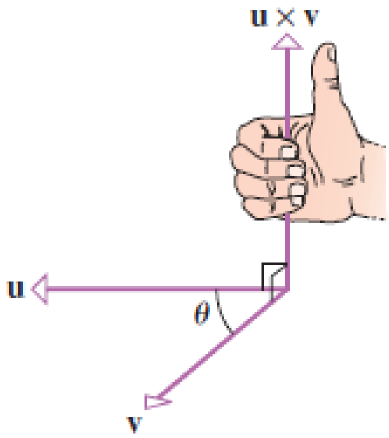
$$\mathbf{u}\times\mathbf{v} = \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1\rangle .$$

From this, we have

$$\begin{aligned} \mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) &= \langle u_1, u_2, u_3\rangle\cdot\langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1\rangle \\ &= u_1(u_2v_3 - u_3v_2) - u_2(u_1v_3 - u_3v_1) + u_3(u_1v_2 - u_2v_1) \\ &= u_1u_2v_3 - u_1u_3v_2 - u_2u_1v_3 + u_2u_3v_1 + u_3u_1v_2 - u_3u_2v_1 \\ &= 0. \end{aligned}$$

Similarly we have $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v})=0$. Hence $\mathbf{u}\times\mathbf{v}$ is a vector which is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. ■

▶ It can be seen that if $\mathbf{u}$ and $\mathbf{v}$ are nonzero and nonparallel vectors, then the direction of $\mathbf{u}\times\mathbf{v}$ relative to $\mathbf{u}$ and $\mathbf{v}$ is determined by a right-hand rule; that is, if the fingers of the right hand are cupped so they curl from $\mathbf{u}$ toward $\mathbf{v}$ in the direction of rotation that takes $\mathbf{u}$ into $\mathbf{v}$ in less than $180°$, then the thumb will point (roughly) in the direction of $\mathbf{u}\times\mathbf{v}$.



The next theorem lists some more important geometric properties of the cross product.

Theorem 2.8

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in 3-space, and let θ be the angle between these vectors when they are positioned so their initial points coincide.

(a) $\|\mathbf{u}\times\mathbf{v}\|=\|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta$

(b) The area A of the parallelogram that has $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides is

$$A=\|\mathbf{u}\times\mathbf{v}\| \tag{22}$$

(c) $\mathbf{u}\times\mathbf{v}=\mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel vectors, that is, if and only if they are scalar multiples of one another.

Proof. (a) We have from (10),

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$

Then,

$$\begin{aligned}
\|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta &= \|\mathbf{u}\|\,\|\mathbf{v}\|\sqrt{\cos^2\theta}\\
&= \|\mathbf{u}\|\,\|\mathbf{v}\|\sqrt{1-\frac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{u}\|^2\,\|\mathbf{v}\|^2}}\\
&= \|\mathbf{u}\|\,\|\mathbf{v}\|\frac{\sqrt{\|\mathbf{u}\|^2\,\|\mathbf{v}\|^2-(\mathbf{u}\cdot\mathbf{v})^2}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\\
&= \sqrt{\|\mathbf{u}\|^2\,\|\mathbf{v}\|^2-(\mathbf{u}\cdot\mathbf{v})^2}\\
&= \sqrt{(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)-(u_1v_1+u_2v_2+u_3v_3)^2} \qquad (*1)
\end{aligned}$$

We have

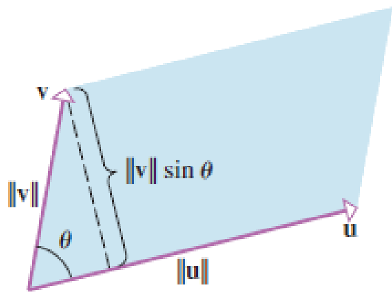
$(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)-(u_1v_1+u_2v_2+u_3v_3)^2=$
$u_1^2v_1^2+u_1^2v_2^2+u_1^2v_3^2+u_2^2v_1^2+u_2^2v_2^2+u_2^2v_3^2+u_3^2v_1^2+u_3^2v_2^2+u_3^2v_3^2-(u_1^2v_1^2+u_2^2v_2^2+u_3^2v_3^2+$
$2u_1v_1u_2v_2+2u_1v_1u_3v_3+2u_2v_2u_3v_3)=(u_1v_2)^2+(u_1v_3)^2+(u_2v_1)^2+(u_2v_3)^2+$
$(u_3v_1)^2+(u_3v_2)^2-2u_1v_1u_2v_2-2u_1v_1u_3v_3-2u_2v_2u_3v_3=(u_1v_2)^2-2u_1v_2u_2v_1+$
$(u_2v_1)^2+(u_1v_3)^2-2u_1v_3u_3v_1+(u_3v_1)^2+(u_2v_3)^2-2u_2v_3u_3v_2+(u_3v_2)^2=$
$(u_1v_2-u_2v_1)^2+(u_1v_3-u_3v_1)^2+(u_2v_3-u_3v_2)^2=\|\mathbf{u}\times\mathbf{v}\|^2$

Substituting in (*1), we get

$$\|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta=\|\mathbf{u}\times\mathbf{v}\|\,.$$

(b) From the figure the parallelogram that has $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides can be viewed as having base $\|\mathbf{u}\|$ and altitude $\|\mathbf{v}\|\sin\theta$. Thus, its area A is

$$A=(\text{base})(\text{altitude})=\|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta=\|\mathbf{u}\times\mathbf{v}\|\,.$$



(c) Since $\mathbf{u}$ and $\mathbf{v}$ are assumed to be nonzero vectors, it follows from part (a) that $\mathbf{u}\times\mathbf{v}=\mathbf{0}$ if and only if $\sin\theta=0$; this is true if and only if $\theta=0$ or $\theta=\pi$ (since $0\le\theta\le\pi$). Geometrically, this means that $\mathbf{u}\times\mathbf{v}=\mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel vectors. ■

Problem 2.21

Find the area of the triangle that is determined by the points $P_1(2,2,0)$, $P_2(-1,0,2)$, and $P_3(0,4,3)$.

Solution. The area A of the triangle is half the area of the parallelogram determined by the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$. But

$$\overrightarrow{P_1P_2}=\langle -3,-2,2\rangle \text{ and } \overrightarrow{P_1P_3}=\langle -2,2,3\rangle\,,$$

so

$$\begin{aligned}\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-3&-2&2\\-2&2&3\end{vmatrix}\\&=(-2\times3-2\times2)\mathbf{i}-(-3\times3-2\times(-2))\mathbf{j}+(3\times2-(-2)\times(-2))\mathbf{k}\\&=(-6-4)\mathbf{i}-(-9+4)\mathbf{j}+(-6-4)\mathbf{k}\\&=-10\mathbf{i}+5\mathbf{j}-10\mathbf{k}\end{aligned}$$

and hence

$$\left\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\right\| = \sqrt{(-10)^2 + 5^2 + (-10)^2} = \sqrt{225} = 15.$$

Hence area of the triangle,

$$A = \frac{1}{2}\left\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\right\| = \frac{15}{2}.$$

SCALAR TRIPLE PRODUCTS

If $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3\rangle$ are vectors in 3-space, then the number

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

is called the scalar triple product of $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$. It can be seen that

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{23}$$

Proof. By (20) and (21)

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= (v_2w_3 - v_3w_2)\mathbf{i} - (v_1w_3 - v_3w_1)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \cdot ((v_2w_3 - v_3w_2)\mathbf{i} - (v_1w_3 - v_3w_1)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}) \\ &= (v_2w_3 - v_3w_2)u_1 - (v_1w_3 - v_3w_1)u_2 + (v_1w_2 - v_2w_1)u_3 \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \end{aligned}$$

Problem 2.22

Find $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

(a) $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}, \mathbf{v} = 4\mathbf{i} + \mathbf{j} - 3\mathbf{k}, \mathbf{w} = \mathbf{j} + 5\mathbf{k}$

(b) $\mathbf{u} = \langle 1, -2, 2\rangle, \mathbf{v} = \langle 0, 3, 2\rangle, \mathbf{w} = \langle -4, 1, -3\rangle$

(c) $\mathbf{u} = \langle 2, 1, 0\rangle, \mathbf{v} = \langle 1, -3, 1\rangle, \mathbf{w} = \langle 4, 0, 1\rangle$

(d) $\mathbf{u} = \mathbf{i}, \mathbf{v} = \mathbf{i} + \mathbf{j}, \mathbf{w} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

Solution. (a) We have

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= \begin{vmatrix} 2 & -3 & 1 \\ 4 & 1 & -3 \\ 0 & 1 & 5 \end{vmatrix} \\ &= 2(1\times 5 - 1\times(-3)) - (-3)(4\times 5 - 0\times(-3)) + 1(4\times 1 - 0\times 1) \\ &= 2(5+3) + 3(20-0) + (4-0) \\ &= 80.\end{aligned}$$

(b) We have

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= \begin{vmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ -4 & 1 & -3 \end{vmatrix} \\ &= 1(3\times(-3) - 1\times 2) - (-2)(0\times(-3) - 2\times(-4)) + 2(0\times 1 - 3\times(-4)) \\ &= (-9-2) + 2(0+8) + 2(0+12) \\ &= 29.\end{aligned}$$

(c), (d) are left as exercise. ■

GEOMETRIC PROPERTIES OF THE SCALAR TRIPLE PRODUCT

Theorem 2.9

Let $\mathbf{u}, \mathbf{v}$, *and* $\mathbf{w}$ *be nonzero vectors in 3-space.*

(a) The volume V *of the parallelepiped that has* $\mathbf{u}, \mathbf{v}$, *and* $\mathbf{w}$ *as adjacent edges is*

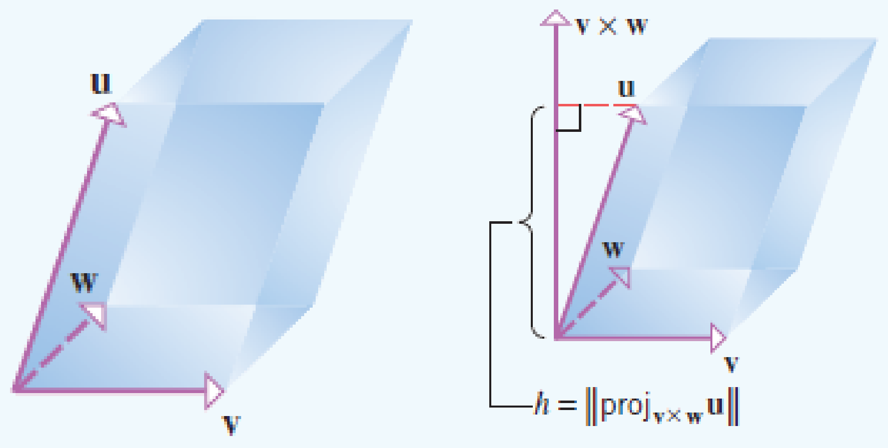
$$V = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| \tag{24}$$

(b) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 0$ *if and only if* $\mathbf{u}, \mathbf{v}$, *and* $\mathbf{w}$ *lie in the same plane.*

Proof. (a) Base of the parallelepiped is the parallelogram with adjacent sides $\mathbf{v}$ and $\mathbf{w}$. So base area is

$$\text{base area} = \|\mathbf{v}\times\mathbf{w}\|\,.$$

Figure 2.17



The height h of the parallelepiped is the length of the orthogonal projection of $\mathbf{u}$ on the vector $\mathbf{v} \times \mathbf{w}$ (since $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, it is along the height of the parallelepiped). Thus

$$\begin{aligned} h &= \left\|\operatorname{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}\right\| \\ &= \frac{|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|}{\|\mathbf{v}\times\mathbf{w}\|^2}\,\|\mathbf{v}\times\mathbf{w}\| \\ &= \frac{|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|}{\|\mathbf{v}\times\mathbf{w}\|}. \end{aligned}$$

Hence volume of the parallelepiped is

$$\begin{aligned} V &= \text{base area } \times \text{ height} \\ &= \|\mathbf{v}\times\mathbf{w}\| \times \frac{|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|}{\|\mathbf{v}\times\mathbf{w}\|} \\ &= |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| \end{aligned}$$

(b) The vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ lie in the same plane if and only if the parallelepiped with these vectors as adjacent sides has volume zero. Thus, from part (a) the vectors lie in the same plane if and only if $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 0$. ■

Problem 2.23

Consider the parallelepiped with adjacent edges $\mathbf{u} = 3\mathbf{i}+2\mathbf{j}+\mathbf{k}$, $\mathbf{v} = \mathbf{i}+\mathbf{j}+2\mathbf{k}$, $\mathbf{w} = \mathbf{i}+3\mathbf{j}+3\mathbf{k}$.

(a) Find the volume.

(b) Find the area of the face determined by $\mathbf{u}$ and $\mathbf{w}$.

(c) Find the angle between $\mathbf{u}$ and the plane containing the face determined by $\mathbf{v}$ and $\mathbf{w}$.

Solution. (a) Volume, $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. We have

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \begin{vmatrix} 3 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{vmatrix} \\
&= 3(3-6) - 2(3-2) + 1(3-1) \\
&= -9 - 2 + 2 \\
&= -9.
\end{aligned}$$

Hence volume, $V = 9$.
(b) Area of the face determined by $\mathbf{u}$ and $\mathbf{w}$ is $\|\mathbf{u} \times \mathbf{w}\|$. We have

$$\begin{aligned}
\mathbf{u} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ 1 & 3 & 3 \end{vmatrix} \\
&= \mathbf{i}(6-3) - \mathbf{j}(9-1) + \mathbf{k}(9-2) \\
&= 3\mathbf{i} - 8\mathbf{j} + 7\mathbf{k}.
\end{aligned}$$

Hence area of the face determined by $\mathbf{u}$ and $\mathbf{w}$ is

$$\|-3\mathbf{i} - \mathbf{j} + 2\mathbf{k}\| = \sqrt{3^2 + (-8)^2 + 7^2} = \sqrt{122}.$$

(c) We know that $\mathbf{v} \times \mathbf{w}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$ and so $\mathbf{v} \times \mathbf{w}$ is perpendicular to the plane containing $\mathbf{v}$ and $\mathbf{w}$. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v} \times \mathbf{w}$. Then the angle between $\mathbf{u}$ and the plane containing the face determined by $\mathbf{v}$ and $\mathbf{w}$ is

$$\pm(90^\circ - \theta).$$

We have

$$\begin{aligned}
\cos\theta &= \frac{\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})}{\|\mathbf{u}\| \, \|\mathbf{u} \times \mathbf{w}\|} \\
&= \frac{-9}{\sqrt{3^2 + 2^2 + 1^2}\sqrt{14}} \\
&= \frac{-9}{\sqrt{14}\sqrt{14}} \\
&= \frac{-9}{14} \\
\Rightarrow \theta &\approx 130.01^\circ
\end{aligned}$$

Hence the angle between $\mathbf{u}$ and the plane containing the face determined by $\mathbf{v}$ and $\mathbf{w}$ is

$$\theta - 90^\circ = 130.01^\circ - 90^\circ = 40.01^\circ.$$ ■

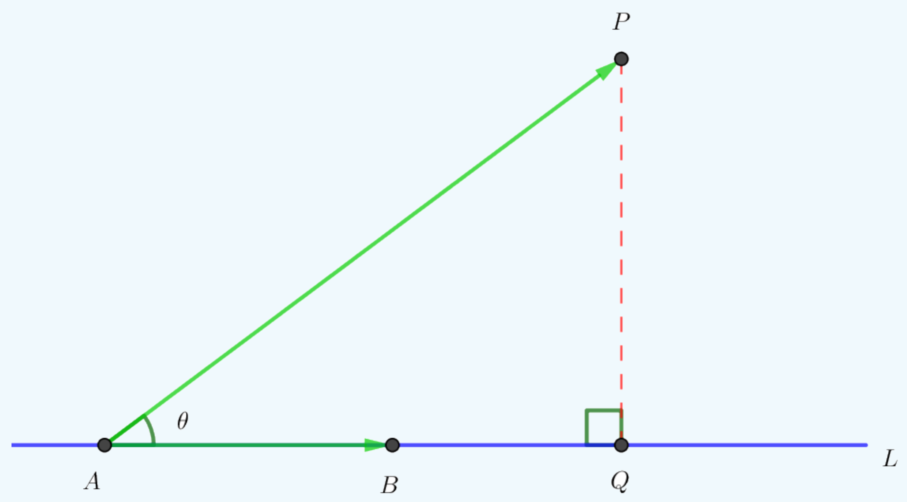
Problem 2.24

Show that in 3-space the distance d from a point P to the line L through points A and B can be expressed as

$$d = \frac{\left\|\overrightarrow{AP} \times \overrightarrow{AB}\right\|}{\left\|\overrightarrow{AB}\right\|}. \tag{25}$$

Solution. Refer the figure:

Figure 2.18



It is clear that

$$d = PQ = \left\|\overrightarrow{AP}\right\| \sin\theta$$

But

$$\overrightarrow{AP} \times \overrightarrow{AB} = \left\|\overrightarrow{AP}\right\| \left\|\overrightarrow{AB}\right\| \sin\theta$$

so that

$$\left\|\overrightarrow{AP} \times \overrightarrow{AB}\right\| = \left\|\overrightarrow{AP}\right\| \left\|\overrightarrow{AB}\right\| |\sin\theta| = \left\|\overrightarrow{AP}\right\| \left\|\overrightarrow{AB}\right\| \sin\theta.$$

(here $\sin\theta > 0$ since $0 < \theta < \pi$). Hence

$$d = \frac{\left\|\overrightarrow{AP} \times \overrightarrow{AB}\right\|}{\left\|\overrightarrow{AB}\right\|}.$$

■

Problem 2.25

Find the distance between the point $P(-3, 1, 2)$ and the line through the points $A(1, 1, 0)$ and $B(-2, 3, -4)$.

Solution. We have

$$\overrightarrow{AP} = (1-(-3))\mathbf{i} + (1-1)\mathbf{j} + (0-2)\mathbf{k} = 4\mathbf{i} - 2\mathbf{k}$$
$$\overrightarrow{AB} = (1-(-2))\mathbf{i} + (1-3)\mathbf{j} + (0-(-4))\mathbf{k} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}.$$

Hence

$$\begin{aligned}
\overrightarrow{AP} \times \overrightarrow{AB} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & -2 \\ 3 & -2 & 4 \end{vmatrix} \\
&= (0 \times 4 - (-2) \times (-2))\mathbf{i} - (4 \times 4 - 3 \times (-2))\mathbf{j} + (4 \times (-2) - 3 \times 0)\mathbf{k} \\
&= (0-4)\mathbf{i} - (16+6)\mathbf{j} + (-8-0)\mathbf{k} \\
&= -4\mathbf{i} - 22\mathbf{j} - 8\mathbf{k}.
\end{aligned}$$

Hence

$$\begin{aligned}
\left\|\overrightarrow{AP} \times \overrightarrow{AB}\right\| &= \sqrt{(-4)^2 + (-22)^2 + (-8)^2} \\
&= \sqrt{564} \\
\left\|\overrightarrow{AB}\right\| &= \sqrt{3^2 + (-2)^2 + 4^2} \\
&= \sqrt{29}.
\end{aligned}$$

Hence required distance is

$$\frac{\left\|\overrightarrow{AP} \times \overrightarrow{AB}\right\|}{\left\|\overrightarrow{AB}\right\|} = \sqrt{\frac{564}{29}}. \quad \blacksquare$$

Problem 2.26

What can you say about the angle between nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ if $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u} \times \mathbf{v}\|$?

Solution. Let θ be the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. By definition,

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta \text{ and } \mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta.$$

Also

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta.$$

Hence

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= \|\mathbf{u} \times \mathbf{v}\| \\
\Rightarrow \cos\theta &= \sin\theta \\
\Rightarrow \theta &= \frac{\pi}{4}.
\end{aligned} \quad \blacksquare$$

Problem 2.27

Show that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in 3-space, then

$$\|\mathbf{u}\times\mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2.$$

Solution. Let θ be the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. Then we have

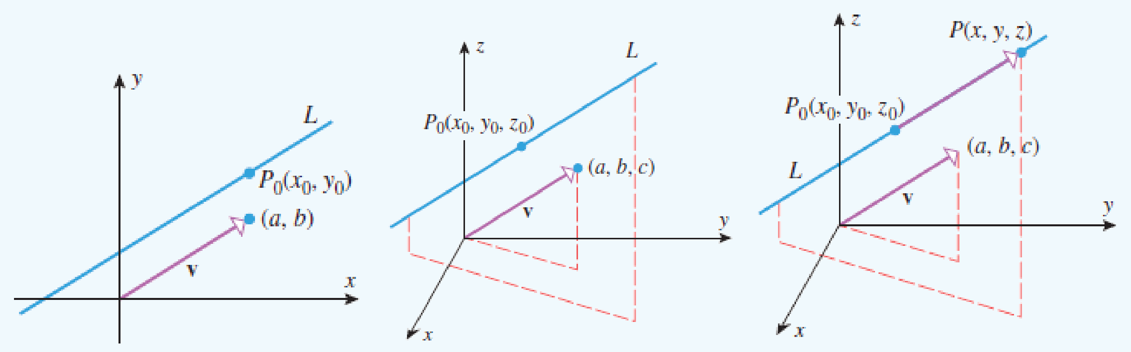
$$\begin{aligned}\|\mathbf{u}\times\mathbf{v}\|^2 &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2\sin^2\theta\\ &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2(1-\cos^2\theta)\\ &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2\|\mathbf{v}\|^2\cos^2\theta\\ &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2.\end{aligned}$$

■

3 PARAMETRIC EQUATIONS OF LINES

We know that there are infinitely many lines passing through a given point on 2-space or 3-space. Similarly there are infinitely many lines parallel to a given line or vector. But there is a unique line passing through a given point and parallel to a given vector in 2-space or 3-space.

Figure 3.1



consider a line L in 3-space that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the nonzero vector $\mathbf{v} = \langle a, b, c\rangle$. Then L consists precisely of those points $P(x, y, z)$ for which the vector $\overrightarrow{PP_0}$ is parallel to $\mathbf{v}$. In other words $\overrightarrow{PP_0}$ is a scalar multiple of $\mathbf{v}$, say

$$\overrightarrow{PP_0} = t\mathbf{v}.$$

That is

$$\langle x - x_0, y - y_0, z - z_0\rangle = \langle ta, tb, tc\rangle$$

which implies that

$$x - x_0 = at, \quad y - y_0 = bt, \quad z - z_0 = ct.$$

Thus parametric equations of L are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

Theorem 3.1

(a) *The line in 2-space that passes through the point $P_0(x_0, y_0)$ and is parallel to the nonzero vector $\mathbf{v} = \langle a, b\rangle = a\mathbf{i} + b\mathbf{j}$ has parametric equations*

$$x = x_0 + at, \quad y = y_0 + bt \tag{26}$$

(b) *The line in 3-space that passes through the point $P_0(x_0, y_0)$ and is parallel to the nonzero vector $\mathbf{v} = \langle a, b\rangle\, c = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ has parametric equations*

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct \tag{27}$$

VECTOR EQUATIONS OF LINES

Representing (26) and (27) in vector form, we get

$$\langle x, y\rangle = \langle x_0 + at, y_0 + bt\rangle$$
$$\langle x, y, z\rangle = \langle x_0 + at, y_0 + bt, z_0 + ct\rangle$$

which shows that

$$\langle x, y\rangle = \langle x_0, y_0\rangle + t\,\langle a, b\rangle$$
$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\,\langle a, b, c\rangle\,.$$

If we write

$$\mathbf{r} = \langle x, y\rangle\,, \quad \mathbf{r}_0 = \langle x_0, y_0\rangle\,, \quad \mathbf{v} = \langle a, b\rangle$$

in 2-space and

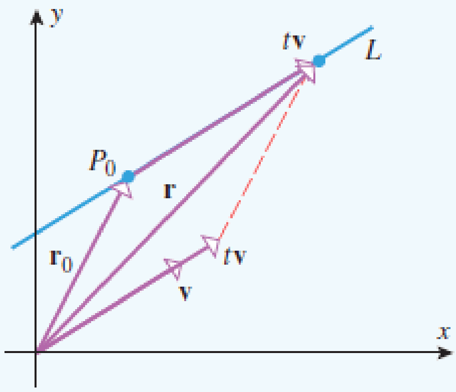
$$\mathbf{r} = \langle x, y, z\rangle\,, \quad \mathbf{r}_0 = \langle x_0, y_0, z_0\rangle\,, \quad \mathbf{v} = \langle a, b, c\rangle$$

in 3-space, we get the vector equation of line as

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} \tag{28}$$

in 2-space or 3-space.

Figure 3.2



$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$

Problem 3.1

Find parametric equations of the line

(a) passing through $(4, 2)$ and parallel to $\mathbf{v} = \langle -1, 5 \rangle$;

(b) passing through $(1, 2, -3)$ and parallel to $\mathbf{v} = 4\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}$;

(c) passing through the origin in 3-space and parallel to $\mathbf{v} = 111$.

Solution. (a) We have $(x_0, y_0) = (4, 2)$ and $a = -1, b = 5$. Hence parametric equations are

$$x = 4 - t, \quad y = 2 + 5t$$

(b) Here $(x_0, y_0, z_0) = (1, 2, -3)$ and $a = 4, b = 5, c = -7$. Hence parametric equations are

$$x = 1 + 4t, \quad y = 2 + 5t, \quad z = -3 - 7t.$$

(c) Here $x_0 = y_0 = z_0 0$ and $a = 1, b = 1, c = 1$. Hence parametric equations are

$$x = t, \quad y = t, \quad z = t.$$

■

Problem 3.2

Find parametric equations describing the line segment joining the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.

Solution. The following are steps to find equation of line through P_1 and P_2:

- Choose any of the points P_1 or P_2 as point on the line. We choose $P_1(2, 4, -1)$.

▶ The line is parallel to the vector $\overrightarrow{P_1P_2} = \langle 5-2, 0-4, 7-(-1)\rangle = \langle 3, -4, 8\rangle$.

Hence the equation of line is

$$x = 2 + 3t, \quad y = 4 - 4t, \quad z = -1 + 8t.$$

From these equations, when $t = 0$ we get P_1 and when $t = 1$, we get P_2. Hence equation of the line segment joining P_1 and P_2 is

$$x = 2 + 3t, \quad y = 4 - 4t, \quad z = -1 + 8t, \quad 0 \le t \le 1.$$ ■

The answer in problem 3.2, can be rewritten using vectors. From the solution,

$$\begin{aligned} \langle x, y, z\rangle &= \langle 2+3t, 4-4t, -1+8t\rangle = \langle 2, 4, -1\rangle + t\langle 3, -4, 8\rangle \\ \Rightarrow \langle x-2, y-4, z+1\rangle &= t\langle 3, -4, 8\rangle, \quad 0 \le t \le 1 \end{aligned}$$

That is the equation is

$$\overrightarrow{P_1P} = t\overrightarrow{P_1P_2}, \quad 0 \le t \le 1$$

where $P(x, y, z)$ is a point on the segment. Generally

Segments

If P is point on the line segment joining the points P_1 and P_2, then the equation of the line segment is

$$\overrightarrow{P_1P} = t\overrightarrow{P_1P_2}, \quad 0 \le t \le 1. \tag{29}$$

Problem 3.3

The parametric equations of a line are

$$x = 2 - t, y = -3 + 5t, z = t$$

(a) Find a point on the line

(b) Determine a vector parallel to the line

Solution. Compare with the (27) we get $(2, -3, 0)$ as a point on the line. Also it is parallel to the vector $= -\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ (since the coefficients of t are $-1, 5, 1$). ■

Problem 3.4

Find parametric equations of the line that satisfies the stated conditions

1. The line through $(-1, 2, 4)$ that is parallel to $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$.
2. The line through $(-2, 0, 5)$ that is parallel to the line given by $x = 1 + 2t, y = 4 - t, z = 6 + 2t$.
3. The line that is tangent to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.
4. Where does the line $\langle x, y\rangle = \langle 4t, 3t\rangle$ intersect the circle $x^2 + y^2 = 25$?
5. Where does the line $x = 1 + 3t, y = 2 - t$ intersect

   (a) the x-axis (b) the y-axis

   (c) the parabola $y = x^2$?

Solution. 1. Clearly $(x_0, y_0, z_0) = (-1, 2, 4)$ and $a = 3, b = -4, c = 1$. So the equation is

$$x = -1 + 3t, \quad y = 2 - 4t, \quad z = 4 + t.$$

2. Clearly $(x_0, y_0, z_0) = (-2, 0, 5)$ and since the given line is parallel to the vector $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, we have $a = 2, b = -1, c = 2$. So the equation is

$$x = -2 + 2t, \quad y = -t, \quad z = 5 + 2t.$$

3. Here we first find slope of the line. It is same as slope of the tangent to the given circle at $(3, -4)$. From the equation $x^2 + y^2 = 25$, we have

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}.$$

Hence slope of the line is

$$-\frac{3}{-4} = \frac{3}{4}.$$

Using the slope, we find a vector parallel to the line. Recall that parallel lines and vector have same slope. Thus a vector parallel to the line is

$$\mathbf{v} = 4\mathbf{i} + 3\mathbf{j}.$$

Since the line passes through $(3, -4)$, the parametric equations are

$$x = 3 + 4t, \quad y = -4 + 3t$$

4. Clearly any point on the line is $x = 4t, y = 3t$. Substituting in the equation of circle we get

$$(4t)^2 + (3t)^2 = 25 \Rightarrow 25t^2 = 25 \Rightarrow t^2 = 1 \Rightarrow \pm 1$$

Hence the line intersect the circle at $(4, 3)$ and $(-4, -3)$.

5. (a) At x-axis, $y = 0$, that is, $2 - t = 0 \Rightarrow t = 2$. Hence the point of intersection with x-axis is

$$x = 1 + 3 \times 2 = 7, y = 0 \Rightarrow (7, 0)$$

(b) At y-axis, $x = 0$, that is, $x = 1 + 3t = 0 \Rightarrow t = -\frac{1}{3}$. Hence the point of intersection with y-axis is

$$x = 0, y = 2 - (-\frac{1}{3}) = 2 + \frac{1}{3} = \frac{7}{3} \Rightarrow \left(0, \frac{7}{3}\right)$$

(c) At the parabola $y = x^2$, we have

$$2 - t = (1 + 3t)^2 \Rightarrow 9t^2 + 7t - 1 = 0.$$

Solving this equation we get

$$t = \frac{-7 \pm \sqrt{85}}{18}.$$

When $t = \frac{-7+\sqrt{85}}{18}$, we get

$$\begin{aligned} x &= 1 + 3 \times \frac{-7 + \sqrt{85}}{18} \\ &= \frac{18 - 21 + 3\sqrt{85}}{18} \\ &= \frac{-3 + 3\sqrt{85}}{18} \\ &= \frac{-1 + \sqrt{85}}{6} \\ y &= 2 - \frac{-7 + \sqrt{85}}{18} \\ &= \frac{36 + 7 - \sqrt{85}}{18} \\ &= \frac{43 - \sqrt{85}}{18} \end{aligned}$$

When $t = \frac{-7-\sqrt{85}}{18}$, we get

$$\begin{aligned} x &= 1 + 3 \times \frac{-7 + \sqrt{85}}{18} \\ &= \frac{18 - 21 - 3\sqrt{85}}{18} \\ &= \frac{-3 - 3\sqrt{85}}{18} \\ &= \frac{-1 - \sqrt{85}}{6} \\ y &= 2 - \frac{-7 - \sqrt{85}}{18} \\ &= \frac{36 + 7 + \sqrt{85}}{18} \\ &= \frac{43 + \sqrt{85}}{18} \end{aligned}$$

Hence points of intersection with the parabola $y = x^2$ are

$$\left(\frac{-1+\sqrt{85}}{6}, \frac{43-\sqrt{85}}{18}\right) \quad \text{and} \quad \left(\frac{-1-\sqrt{85}}{6}, \frac{43+\sqrt{85}}{18}\right). \qquad \blacksquare$$

Problem 3.5

Find the intersections of the lines with the xy-plane, the xz-plane, and the yz-plane.

(a) $x = -1 + 2t, y = 3 + t, z = 4 - t$

(b) $x = -2, y = 4 + 2t, z = -3 + t$

Solution. (a) On xy-plane, $z = 0$, so that

$$4 - t = 0 \Rightarrow t = 4.$$

Then

$$x = -1 + 2 \times 4 = 7, \quad y = 3 + 4 = 7, \quad z = 0.$$

Hence point of intersection with xy-plane is

$$(7, 7, 0)$$

On xz-plane, $y = 0$, so that

$$3 + t = 0 \Rightarrow t = -3.$$

Then

$$x = -1 + 2 \times (-3) = -7, \quad y = 0, z = 4 - (-3) = 7.$$

Hence point of intersection with xz-plane is

$$(-7, 0, 7)$$

On yz-plane, $x = 0$, so that

$$-1 + 2t = 0 \Rightarrow t = \frac{1}{2}.$$

Then

$$x = 0, \quad y = 3 + \frac{1}{2} = \frac{7}{2}, \quad z = 4 - \frac{1}{2} = \frac{7}{2}.$$

Hence point of intersection with yz-plane is

$$\left(0, \frac{7}{2}, \frac{7}{2}\right).$$

(b) Left as exercise. (Note that on yz-plane, $x = 0$. Here $x = -2$, which shows that the line does not intersect the yz-plane.) $\blacksquare$

Problem 3.6

Where does the line $x = 1 + t, y = 3 - t, z = 2t$ intersect the cylinder $x^2 + y^2 = 16$?

Solution. Substituting values of x, y in the equation of cylinder we get

$$\begin{aligned} (1+t)^2 + (3-t)^2 &= 16 \\ \Rightarrow 1 + 2t + t^2 + 9 - 6t + t^2 &= 16 \\ \Rightarrow 2t^2 - 4t - 6 &= 0 \\ \Rightarrow t^2 - 2t - 3 &= 0 \\ \Rightarrow (t+1)(t-3) &= 0 \Rightarrow t = -1, 3. \end{aligned}$$

Hence points of intersection are

$$(1-1, 3-(-1), 2(-1)) = (0, 4, -2)$$

and

$$(1+3, 3-3, 2 \times 3) = (4, 0, 6).$$

■

Problem 3.7

Where does the line $x = 2 - t, y = 3t, z = -1 + 2t$ intersect the plane $2y + 3z = 6$?

Solution. Left as exercise. (Ans. $(\frac{5}{4}, \frac{9}{4}, \frac{1}{2})$). ■

Problem 3.8

Let L_1 and L_2 be the lines

$$\begin{aligned} L_1 : x &= 1 + 4t, & y &= 5 - 4t, & z &= -1 + 5t \\ L_2 : x &= 2 + 8t, & y &= 4 - 3t, & z &= 5 + t \end{aligned}$$

(a) Are the lines parallel?

(b) Do the lines intersect?

Solution. (a) L_1 is parallel to the vector

$$\mathbf{u}_1 = \langle 4, -4, 5 \rangle$$

and L_2 is parallel to the vector

$$\mathbf{u}_2 = \langle 8, -3, 1 \rangle.$$

Clearly the vectors are not parallel, since $\mathbf{u}_2$ is not a scalar multiple of $\mathbf{u}_1$. (Here x component of $\mathbf{u}_1$ is 4 and x component of $\mathbf{u}_2$ is 8, moreover $8 = 4 \times 2$. But for y component of $\mathbf{u}_1$ is -4 and y component of $\mathbf{u}_2$ is $-3 \neq -4 \times 2$.)

Hence the lines are not parallel.

(b) For L_1 and L_2 to intersect at some point (x_0, y_0, z_0) these coordinates would have to satisfy the equations of both lines. That is

$$x_0 = 1 + 4t_1, \quad y_0 = 5 - 4t_1, \quad z_0 = -1 + 5t_1$$

and

$$x_0 = 2 + 8t_2, \quad y_0 = 4 - 3t_2, \quad z_0 = 5 + t_2$$

for some values t_1 and t_2 of t. From these we get

$$\begin{aligned} 1 + 4t_1 &= 2 + 8t_2 \\ 5 - 4t_1 &= 4 - 3t_2 \\ -1 + 5t_1 &= 5 + t_2 \end{aligned}$$

We solve the first two equations to find t_1 and t_2. If the obtained values satisfy the third equation, the lines intersect, do not intersect otherwise.

Adding the first two equations, we get

$$6 = 6 + 5t_2 \Rightarrow t_2 = 0.$$

Substituting in first equation, we get,

$$1 + 4t_1 = 2 \Rightarrow t_1 = \frac{1}{4}.$$

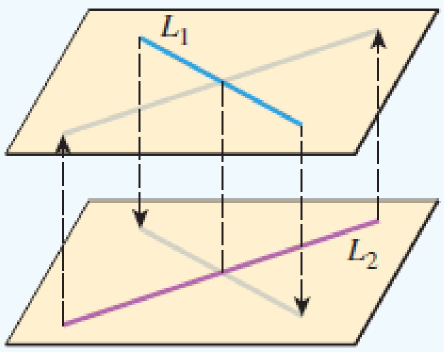
Clearly these values do not satisfy the third equation, since third equation becomes,

$$\begin{aligned} -1 + 5 \times \frac{1}{4} &= 5 + 0 \\ \Rightarrow \frac{1}{4} &= 5. \end{aligned}$$

Hence the lines do not intersect. ■

▶ Two lines in 3-space that are not parallel and do not intersect are called ***skew lines***.

Figure 3.3



Problem 3.9

Show that the lines L_1 and L_2 intersect, and find their point of intersection.

(a) $L_1 : x+1 = 4t, \quad y-3 = t, \quad z-1 = 0$
$L_2 : x+13 = 12t, \quad y-1 = 6t, \quad z-2 = 3t$

(b) $L_1 : x = 2+t, \quad y = 2+3t, \quad z = 3+t$
$L_2 : x = 2+t, \quad y = 3+4t, \quad z = 4+2t$

Solution. (a) The given equations are

$$L_1 : x = -1+4t, \quad y = 3+t, \quad z = 1$$
$$L_2 : x = -13+12t, \quad y = 1+6t, \quad z = 2+3t.$$

Let (a, b, c) be the point of intersection. Then we get

$$a = -1+4t_1, \qquad b = 3+t_1, \qquad c = 1$$
$$a = -13+12t_2, \qquad b = 1+6t_2, \qquad c = 2+3t_2$$

for some values t_1, t_2 of t. Which gives

$$1+4t_1 = -13+12t_2$$
$$3+t_1 = 1+6t_2$$
$$2+3t_2 = 1.$$

From the third equation

$$3t_2 = -1 \Rightarrow t_2 = -\frac{1}{3}.$$

Substituting in the second equation, we get

$$3+t_1 = 1+6\times\left(-\frac{1}{3}\right) \Longrightarrow t_1 = -3+1-2 = -4.$$

Using these values first equation becomes

$$-1+4\times(-4) = -13+12\times\left(-\frac{1}{3}\right)$$
$$\Rightarrow -1-16 = -13-4$$
$$\Rightarrow -17 = -17$$

that is, $t_1 = -4$ and $t_2 = -\frac{1}{3}$ satisfies all equations so that the lines intersect at the point $t = t_1 = -4$ in L_1 (or $t = t_2 = -\frac{1}{3}$ in L_2.) When $t = t_1 = -4$, from L_1, we have

$$a = -1+\times(-4) = -17, \quad b = 3-4 = -1, \quad c = 1.$$

Hence the point of intersection is

$$(-17, -1, 1).$$

Verify that same point is obtained from L_2 when $t = t_2 = -\frac{1}{3}$.
(b) Left as exercise. ■

Problem 3.10

1. Show that the lines L_1 and L_2 are skew.

   (a) $L_1 : x = 1 + 7t, \quad y = 3 + t, \quad z = 5 - 3t$
   $L_2 : x = 4 - t, \quad y = 6, \quad z = 7 + 2t$

   (b) $L_1 : x = 2 + 8t, \quad y = 6 - 8t, \quad z = 10t$
   $L_2 : x = 3 + 8t, \quad y = 5 - 3t, \quad z = 6 + t$

2. Determine whether the lines L_1 and L_2 are parallel.

   (a) $L_1 : x = 3 - 2t, \quad y = 4 + t, \quad z = 6 - t$
   $L_2 : x = 5 - 4t, \quad y = -2 + 2t, \quad z = 7 - 2t$

   (b) $L_1 : x = 5 + 3t, \quad y = 4 - 2t, \quad z = -2 + 3t$
   $L_2 : x = -1 + 9t, \quad y = 5 - 6t, \quad z = 3 + 8t$

3. Determine whether the points P_1, P_2, and P_3 lie on the same line.

   (a) $P_1(6, 9, 7), P_2(9, 2, 0), P_3(0, -5, -3)$

   (b) $P_1(1, 0, 1), P_2(3, -4, -3), P_3(4, -6, -5)$

Solution. The problems 1 and 2 are left as exercise. (Hint. for 2.(a), L_1 is parallel to the vector $\mathbf{v}_1 = \langle -2, 1, -1\rangle$ and L_2 is parallel to the vector $\mathbf{v}_2 = \langle -4, 2, -2\rangle = 2v_1$.)
3. We determine the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_2P_3}$. If they are parallel, then since they have one common end, the points will lie on a line, otherwise not.
(a) We have

$$\overrightarrow{P_1P_2} = \langle 9 - 6, 2 - 9, 0 - 7\rangle = \langle 3, -7, -7\rangle$$

and

$$\overrightarrow{P_2P_3} = \langle 0 - 9, -5 - 2, -3 - 0\rangle = \langle -9, -7, -3\rangle .$$

The vectors are not parallel. So the points do not lie on a line.
(b) Left as exercise. ■

Problem 3.11

Show that the lines L_1 and L_2 are the same.

(a) $L_1 : x = 1 + 3t, y = -2 + t, z = 2t$
$L_2 : x = 4 - 6t, y = -1 - 2t, z = 2 - 4t$

(b) $L_1 : x = 3 - t, y = 1 + 2t$
$L_2 : x = -1 + 3t, y = 9 - 6t$

Solution. In each case we show that the lines have a common point and they are parallel.
(a) When $t = 0$, we see that $(1, -2, 0)$ is a point on L_1. For L_2, the x coordinate of a point is 1 when,

$$4 - 6t = 1 \Rightarrow -6t = -3 \Rightarrow t = \frac{1}{2}.$$

Now for $t = \frac{1}{2}$, in L_2,

$$y = -1 - 2 \times \frac{1}{2} = -1 - 1 = -2 \quad \text{and} \quad z = 2 - 4 \times \frac{1}{2} = 2 - 2 = 0.$$

Hence L_2 also passes through $(1, -2, 0)$.

The line L_1 is parallel to the vector $\mathbf{v}_1 = \langle 3, 1, 2 \rangle$ and L_2 is parallel to the vector $\mathbf{v}_2 = \langle -6, -2, -4 \rangle = -2 \langle 3, 1, 2 \rangle = -2\mathbf{v}_1$. Hence L_1 and L_2 are parallel.

Since L_1 and L_2 are parallel and they pass through $(1, -2, 0)$, they are same.
(b) Left as exercise. ■

Problem 3.12

Describe the line segment represented by the vector equation.

(a) $\langle x, y, z \rangle = \langle -2, 1, 4 \rangle + t \langle 3, 0, -1 \rangle, \quad (0 \le t \le 3)$

(b) $\langle x, y \rangle = \langle 1, 0 \rangle + t \langle -2, 3 \rangle, \quad (0 \le t \le 2)$

(c) Find the point on the line segment joining $P_1(1, 4, -3)$ and $P_2(1, 5, -1)$ that is $\frac{2}{3}$ of the way from P_1 to P_2.

(d) Find the point on the line segment joining $P_1(3, 6)$ and $P_2(8, -4)$ that is $\frac{2}{5}$ of the way from P_1 to P_2.

Solution. (a) When $t = 0$ we get $(x, y, z) = (-2, 1, 4)$ and when $t = 3$ we have $(x, y, z) = (-2, 1, 4) + 3(3, 0, -1) = (7, 1, 1)$. That is the given equation is a line segment joining $(-2, 1, 4)$ and $(7, 1, 1)$.
(b) Left as exercise.

(c) Suppose $P(x, y, z)$ is a point on the segment $\frac{2}{3}$ units of the way from P_1 to P_2. Then from (29), we have

$$\overrightarrow{PP_1} = \frac{2}{3}\overrightarrow{P_1P_2}.$$

That is

$$\begin{aligned}
\langle x-1, y-4, z-(-3)\rangle &= \frac{2}{3}\langle 1-1, 5-4, -1-(-3)\rangle \\
&= \frac{2}{3}\langle 0, 1, 2\rangle \\
&= \left\langle 0, \frac{2}{3}, \frac{4}{3}\right\rangle \\
\Rightarrow \langle x, y, z\rangle &= \langle 1, 4, -3\rangle + \left\langle 0, \frac{2}{3}, \frac{4}{3}\right\rangle \\
&= \left\langle 1, 4+\frac{2}{3}, -3+\frac{4}{3}\right\rangle \\
&= \left\langle 1, \frac{14}{3}, -\frac{5}{3}\right\rangle.
\end{aligned}$$

Hence the point is $\left(1, \frac{14}{3}, -\frac{5}{3}\right)$.

(d) Left as exercise. ■

Problem 3.13

Show that the lines L_1 and L_2 are parallel, and find the distance between them.

(a) $L_1 : x = 2 - t, \quad y = 2t, \quad z = 1 + t$
$L_2 : x = 1 + 2t, \quad y = 3 - 4t, \quad z = 5 - 2t$

(b) $L_1 : x = 2t, \quad y = 3 + 4t, \quad z = 2 - 6t$
$L_2 : x = 1 + 3t, \quad y = 6t, \quad z = -9t$

Solution. (a) The line L_1 is parallel to the vector $\mathbf{v}_1 = \langle -1, 2, 1\rangle$ and L_2 is parallel to the vector $\mathbf{v}_2 = \langle 2, -4, -2\rangle$. Clearly

$$\mathbf{v}_2 = -2\mathbf{v}_1.$$

Hence the lines are parallel. We use (25) to find the distance between the lines.

Method: Take a point say P on one line and two points say A, B on the other line. Then by (25), distance d between the lines is

$$d = \frac{\left\|\overrightarrow{AP} \times \overrightarrow{AB}\right\|}{\left\|\overrightarrow{AB}\right\|}.$$

Consider L_1. Clearly $P(2, 0, 1)$ is a point on L_1. Similarly we choose $A(1, 3, 5)$ on L_2. For B, put a value for t in L_2. We choose $t = 1$, then from L_2 we have $B = (3, -1, 3)$. Then

$$\overrightarrow{AP} = \langle 1, -3, -4\rangle \quad \text{and} \quad \overrightarrow{AB} = \langle 2, -4, -2\rangle$$

and

$$\overrightarrow{AP} \times \overrightarrow{AB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & -4 \\ 2 & -4 & -2 \end{vmatrix}$$
$$= (6-16)\mathbf{i} - (-2+8)\mathbf{j} + (-4+6)\mathbf{k}$$
$$= -10\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}.$$

So that

$$\left\|\overrightarrow{AP} \times \overrightarrow{AB}\right\| = \sqrt{100+36+4}$$
$$= \sqrt{140}$$
$$= 2\sqrt{35}$$
$$\left\|\overrightarrow{AB}\right\| = \sqrt{4+16+4}$$
$$= \sqrt{24}$$
$$= 2\sqrt{6}.$$

Hence

$$d = \frac{2\sqrt{35}}{2\sqrt{6}}$$
$$= \sqrt{\frac{35}{6}}.$$

(b) Left as exercise. ■

Problem 3.14

(a) Find parametric equations for the line through the points (x_0, y_0, z_0) and (x_1, y_1, z_1).

(b) Find parametric equations for the line through the point (x_1, y_1, z_1) and parallel to the line $x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$

Solution. (a) We need a point on the line and a vector parallel to the line. Here $A(x_0, y_0, z_0)$ and $B(x_1, y_1, z_1)$ are points on the line. We choose A. Clearly the line is parallel to $\overrightarrow{AB} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$. Hence a parametric equation of the line is

$$x = x_0 + (x_1 - x_0)t, \quad y = y_0 + (y_1 - y_0)t, \quad z = z_0 + (z_1 - z_0). \tag{30}$$

Find another parametric equations of the line.

(b) Left as exercise. ■

Problem 3.15

Let L be the line that passes through the point (x_0, y_0, z_0) and is parallel to the vector $\mathbf{v} = \langle a, b, c\rangle$, where a, b, and c are nonzero. Show that a point (x, y, z) lies on the line L if and only if

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \tag{31}$$

Solution. From the given information, parametric equations of the line are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

From these we have

$$x - x_0 = at, \quad y - y_0 = bt, \quad z - z_0 = ct.$$

That is

$$\frac{x - x_0}{a} = t, \quad \frac{y - y_0}{b} = t \quad \frac{z - z_0}{c} = t.$$

Hence we get

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

as the equation of the line ■

▶ These equations (31), which are called the ***symmetric equations*** of L, provide a nonparametric representation of L.

Problem 3.16

(a) Describe the line whose symmetric equations are

$$\frac{x - 1}{2} = \frac{y + 3}{4} = z - 5.$$

(b) Find parametric equations for the line in part (a).

Solution. (a) The equation can be rewritten as

$$\frac{x - 1}{2} = \frac{y + 3}{4} = \frac{z - 5}{1}.$$

Comparing with (31), we see that the line passes through the point $(1, -3, 5)$ and it is parallel to the vector $2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$.
(b) Parametric equations is

$$x = 1 + 2t, \quad y = -3 + 4t, \quad z = 5 + t.$$ ■

Problem 3.17

(a) Describe the line whose symmetric equations are

$$\frac{x+\frac{3}{2}}{4}=\frac{z-7}{-2}, \quad y=2.$$

(b) Find parametric equations for the line in part (a).

Solution. (a) The equation can be rewritten as

$$\frac{x+\frac{3}{2}}{4}=\frac{z-7}{-2}, \quad y=2+0t.$$

Comparing with (31), we see that the line passes through the point $\left(-\frac{3}{2},2,7\right)$ and it is parallel to the vector $4\mathbf{i}-2\mathbf{k}$.

(b) Parametric equations is

$$x=-\frac{3}{2}+4t, \quad y=2, \quad z=7-2t. \qquad \blacksquare$$

Problem 3.18

Consider the lines L_1 and L_2 whose symmetric equations are

$$L_1: \frac{x-1}{2}=\frac{y+\frac{3}{2}}{1}=\frac{z+1}{2}$$
$$L_2: \frac{x-4}{-1}=\frac{y-3}{-2}=\frac{z+4}{2}$$

(a) Are L_1 and L_2 parallel? Perpendicular?

(b) Find parametric equations for L_1 and L_2.

(c) Do L_1 and L_2 intersect? If so, where?

Solution. Lines L_1 and L_2 are parallel to the vectors $\mathbf{v}_1=\langle 2,1,2\rangle$ and $\mathbf{v}_2=\langle -1,-2,2\rangle$ respectively.

(a) Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of each other, the lines are not parallel. We have

$$\mathbf{v}_1\cdot\mathbf{v}_2=2(-1)+1(-2)+2\times 2=0$$

so that the lines are perpendicular.

(b) L_1: Since $\left(1,-\frac{3}{2},-1\right)$ is a point on the line, parametric equations are

$$x=1+2t, \quad y=-\frac{3}{2}+t, \quad z=-1+2t.$$

L_2: Since $(4,3,-4)$ is a point on the line, parametric equations are

$$x=4-t, \quad y=3-2t, \quad z=-4+2t.$$

(c) Suppose the lines intersect at (x_0, y_0, z_0). Then from the parametric equations,

$$x_0 = 1 + 2t_1, \qquad y_0 = -\frac{3}{2} + t_1, \qquad z_0 = -1 + 2t_1$$
$$x_0 = 4 - t_2, \qquad y_0 = 3 - 2t_2, \qquad z_0 = -4 + 2t_2$$

for some t_1 and t_2. Then we get

$$\begin{aligned} 1 + 2t_1 &= 4 - t_2 \\ -\frac{3}{2} + t_1 &= 3 - 2t_2 \\ -1 + 2t_1 &= -4 + 2t_2 \end{aligned}$$

From the first and third equations, we get

$$2 = 8 - 3t_2 \Rightarrow 3t_2 = 6 \Rightarrow t_2 = 2.$$

Substituting in the first, we get

$$1 + 2t_1 = 2 \Rightarrow t_1 = \frac{1}{2}.$$

Substituting in the second equation we get

$$-\frac{3}{2} + \frac{1}{2} = 3 - 2 \Rightarrow -1 = -1.$$

Hence the lines intersect and the point of intersection is

$$(x_0, y_0, z_0) = (4 - 2, 3 - 2 \times 2, -4 + 2 \times 2) = (2, -1, 0). \qquad ■$$

Problem 3.19

1. Let L_1 and L_2 be the lines whose parametric equations are

$$L_1 : x = 1 + 2t, \quad y = 2 - t, \quad z = 4 - 2t$$
$$L_2 : x = 9 + t, \quad y = 5 + 3t, \quad z = -4 - t$$

   (a) Show that L_1 and L_2 intersect at the point $(7, -1, -2)$.

   (b) Find, to the nearest degree, the acute angle between L_1 and L_2 at their intersection.

   (c) Find parametric equations for the line that is perpendicular to L_1 and L_2 and passes through their point of intersection.

2. Let L_1 and L_2 be the lines whose parametric equations are

$$L_1 : x = 4t, \quad y = 1 - 2t, \quad z = 2 + 2t$$
$$L_2 : x = 1 + t, \quad y = 1 - t, \quad z = -1 + 4t$$

   (a) Show that L_1 and L_2 intersect at the point $(2, 0, 3)$.

   (b) Find, to the nearest degree, the acute angle between L_1 and L_2 at their intersection.

   (c) Find parametric equations for the line that is perpendicular to L_1 and L_2 and passes through their point of intersection.

Solution. 1. (a) Left as exercise. (Hint put $t = 3$ in the first. Find a similar value of t for second line, so that $(7, -1, -2)$ lies on both lines.)

Also show that the lines are not parallel.

(b) L_1 is parallel to the vector $\mathbf{v}_1 = \langle 2, -1, -2 \rangle$ and L_2 is parallel to the vector $\mathbf{v}_2 = \langle 1, 3, -1 \rangle$. Let θ be the acute angle between L_1 and L_2. It is same as the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$. Then

$$\begin{aligned}
\cos\theta &= \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\| \, \|\mathbf{v}_2\|} \\
&= \frac{(2 \times 1) + (-1 \times 3) + (-2 \times -1)}{\sqrt{2^2 + (-1)^2 + (-2)^2}\sqrt{1^2 + 3^2 + (-1)^2}} \\
&= \frac{1}{\sqrt{9}\sqrt{11}} \\
&= \frac{1}{3\sqrt{11}} \\
\Rightarrow \theta &\approx 84.23^\circ.
\end{aligned}$$

(c) The line L perpendicular to L_1 and L_2 is parallel to the vector $\mathbf{v} = \mathbf{v}_1 \times \mathbf{v}_2$.

We have

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -2 \\ 1 & 3 & -1 \end{vmatrix}$$
$$= (1+6)\mathbf{i} - (-2+2)\mathbf{j} + (6+1)\mathbf{k}$$
$$= 7\mathbf{i} + 7\mathbf{k}$$
$$\Rightarrow \mathbf{v} = 7\mathbf{i} + 7\mathbf{k}.$$

Since L passes through $(7, -1, -2)$, the equation of L is

$$x = 7 + 7t, \quad y = -1 + 0t = -1, \quad z = -2 + 7t.$$

2. Left as exercise.

■

Problem 3.20

Find parametric equations of the line that contains the point P and intersects the line L at a right angle, and find the distance between P and L.

(a) $P(0, 2, 1)$

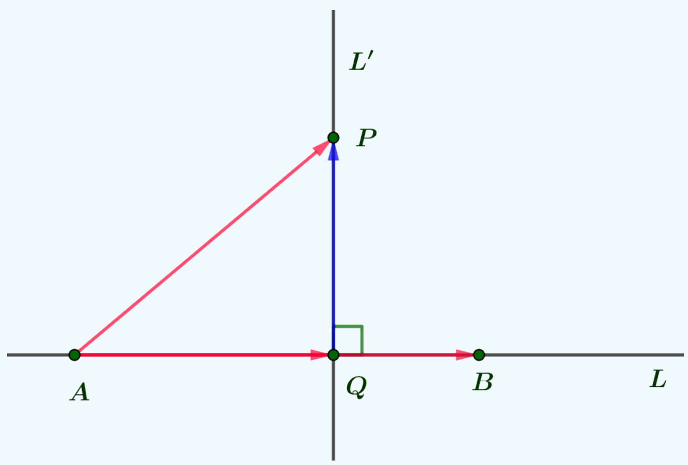
$L : x = 2t, \quad y = 1 - t, \quad z = 2 + t$

(b) $P(3, 1, -2)$

$L : x = -2 + 2t, \quad y = 4 + 2t, \quad z = 2 + t$

Solution. (a) Let the required line be L'. Given that $P(0, 2, 1)$ is a point on the line. Now we need a vector parallel to L'. Let $\mathbf{v}$ be a vector parallel to L'.

Figure 3.4



Choose points A, B on the line L. From the figure it is clear that $\mathbf{v}$ is parallel to $\overrightarrow{QP}$ which the vector component of $\overrightarrow{AP}$ orthogonal to $\overrightarrow{AB}$. To find $\overrightarrow{QP}$ we first find $\overrightarrow{AQ}$ which is the projection of $\overrightarrow{AP}$ on $\overrightarrow{AB}$.

Put $t = 0, 1$ in the equation of L, we get $A(0,1,2), B(2,0,3)$ as points on L. Then

$$\overrightarrow{AP} = \langle 0, 1, -1\rangle \quad \text{and} \quad \overrightarrow{AB} = \langle 2, -1, 1\rangle .$$

Now

$$\overrightarrow{AQ} = \text{proj}_{\overrightarrow{AB}}\overrightarrow{AP} = \left(\frac{\overrightarrow{AP}\cdot\overrightarrow{AB}}{\left\|\overrightarrow{AB}\right\|^2}\right)\overrightarrow{AB}$$

where

$$\overrightarrow{AP}\cdot\overrightarrow{AB} = 0 - 1 - 1 = -2 \quad \text{and} \quad \left\|\overrightarrow{AB}\right\|^2 = 2^2 + (-1)^2 + 1^2 = 6.$$

Hence

$$\overrightarrow{AQ} = \frac{-2}{6}\langle 2, -1, 1\rangle = -\frac{1}{3}\langle 2, -1, 1\rangle = \left\langle -\frac{2}{3}, \frac{1}{3}, -\frac{1}{3}\right\rangle$$

and so

$$\begin{aligned}\overrightarrow{QP} &= \overrightarrow{AP} - \overrightarrow{AQ}\\ &= \langle 0, 1, -1\rangle - \left\langle -\frac{2}{3}, \frac{1}{3}, -\frac{1}{3}\right\rangle\\ &= \left\langle \frac{2}{3}, \frac{2}{3}, -\frac{2}{3}\right\rangle\\ &= \frac{2}{3}\langle 1, 1, -1\rangle .\end{aligned}$$

Since $\mathbf{v}$ is parallel to $\overrightarrow{QP}$, we can choose

$$\mathbf{v} = \langle 1, 1, -1\rangle$$

is parallel to L'. Thus parametric equations of L' are

$$x = 0 + t = t, \quad y = 2 + t, \quad z = 1 - t.$$

We next find the distance d of P from L. Using (25), we see that

$$d = \frac{\left\|\overrightarrow{AP}\times\overrightarrow{AB}\right\|}{\left\|\overrightarrow{AB}\right\|}.$$

Rest is left as exercise.

(b) Left as exercise. ■

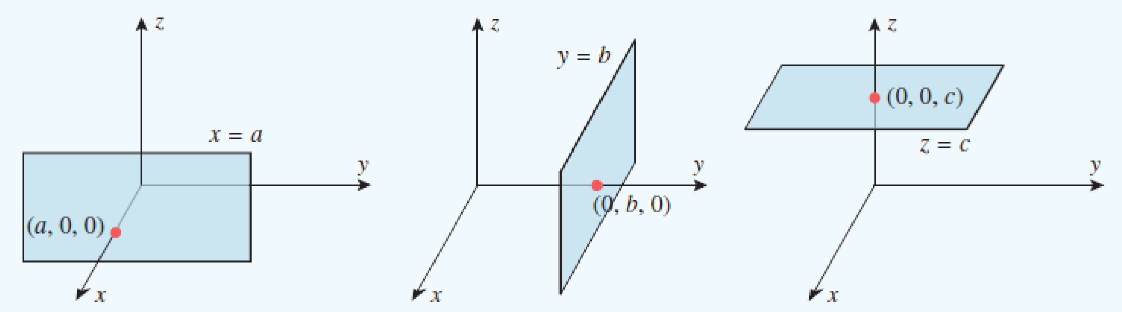
4 PLANES IN 3-SPACE

We know that there are infinitely many planes passing through a given point or a given line on 3-space. Similarly there are infinitely many planes parallel to a given line or vector and infinitely many planes parallel to a given plane. Also there are infinitely many planes perpendicular to a given line or vector and infinitely many planes perpendicular to a given plane. But

- there is a unique plane passing through a given point and parallel to a given vector
- there is a unique plane passing through a given point and perpendicular to a given vector.
- there is a unique plane passing through a given point and parallel to a given plane.

We have seen that the points of the form (a, y, z) where a is a constant lie on the plane parallel to yz-plane and passing through $(a, 0, 0)$. The equation of such a plane is given by $x = a$. Similarly, the graph of $y = b$ is the plane through $(0, b, 0)$ that is parallel to the xz-plane, and the graph of $z = c$ is the plane through $(0, 0, c)$ that is parallel to the xy-plane.

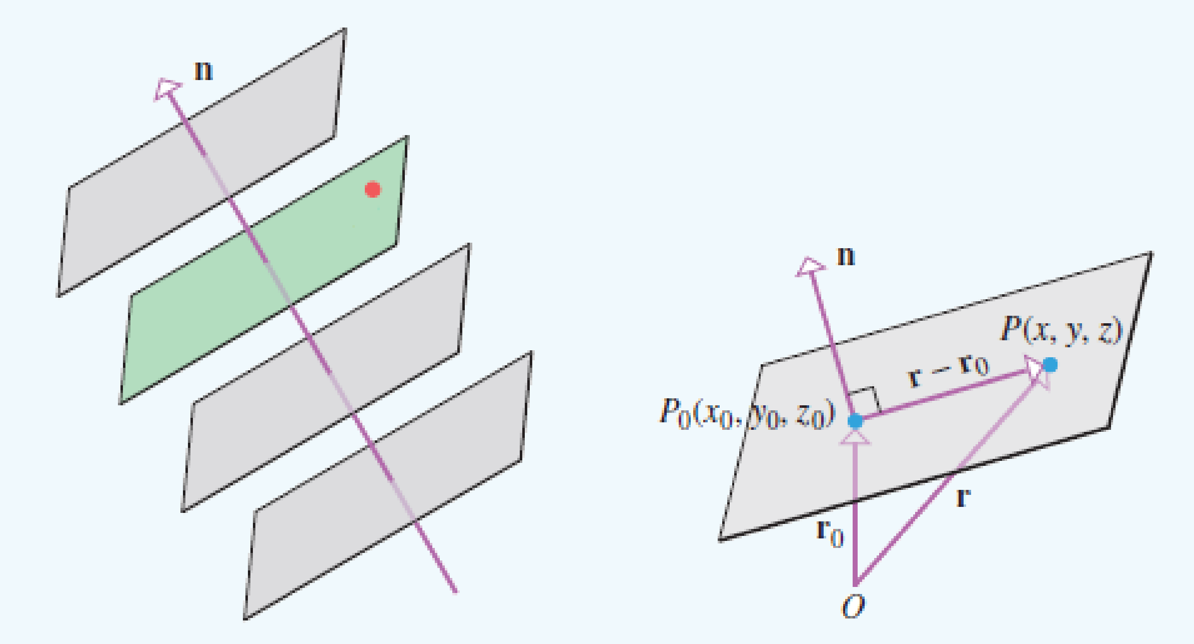
Figure 4.1



PLANES DETERMINED BY A POINT AND A NORMAL VECTOR

Suppose a given plane passes through a point $P_0(x_0, y_0, z_0)$ and perpendicular to a vector $\mathbf{n} = \langle a, b, c\rangle$. A vector perpendicular to a plane is called a ***normal*** to the plane. We derive equation of the plane as follows.

Figure 4.2



Consider a point $P(x, y, z)$ on the plane. Define the vectors $\mathbf{r}$ and $\mathbf{r}_0$ as

$$\mathbf{r} = \langle x, y, z \rangle \text{ and } \mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle .$$

Then

$$\mathbf{r} - \mathbf{r}_0 = \langle x - x_0, y - y_0, z - z_0 \rangle$$

lies on the plane so that it is perpendicular to the normal vector $\mathbf{n}$. Hence we have

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \tag{32}$$

is the equation of the plane. Equivalently we have

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0 \tag{33}$$

or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{34}$$

are equations of the same plane. This is called the ***point-normal form*** of the equation of a plane.

Problem 4.1

Find an equation of the plane that passes through the point P and has the vector $\mathbf{n}$ as a normal.

(a) $P(2,6,1); \quad \mathbf{n} = \langle 1,4,2\rangle$

(b) $P(-1,-1,2); \quad \mathbf{n} = \langle -1,7,6\rangle$

(c) $P(1,0,0); \quad \mathbf{n} = \langle 0,0,1\rangle$

(d) $P(0,0,0); \quad \mathbf{n} = \langle 2,-3,-4\rangle$

Solution. (a) Here $(x_0, y_0, z_0) = (2,6,1)$ and $\langle a,b,c\rangle = \langle 1,4,2\rangle$. Since the equation is

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

we have

$$1(x-2) + 4(y-6) + 2(z-1) = 0 \Rightarrow x - 2 + 4y - 24 + 2z - 2 = 0$$

which becomes

$$x + 4y + 2z - 28 = 0.$$

(b)-(d) are left as exercise. ■

Problem 4.2

Find an equation of the plane that passes through the given points.

(a) $(-2,1,1), (0,2,3)$, and $(1,0,-1)$

(b) $(3,2,1), (2,1,-1)$, and $(-1,3,2)$

Solution. Method : Consider three points (non-collinear) $P_1(x_1,y_1,z_1), P_2(x_2,y_2,z_2)$ and $P_3(x_3,y_3,z_3)$, there is a unique plane containing them. To find an equation of that plane, we need the following

- a point on the plane: choose any one of the points P_1, P_2, P_3,
- a normal to the plane: note that any vector normal to the plane will be perpendicular to the vectors lying on the plane. We choose two vectors using P_1, P_2, P_3, say $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$. We know that $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ is a vector perpendicular to both $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$, so that it will be parallel to the normal. We can choose $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ as a normal vector.

(a) Let $P_1(-2,1,1), P_2(0,2,3)$, and $P_3(1,0,-1)$ be the points. We choose $P_1(-2,1,1)$ as a point on the plane.

Consider

$$\begin{aligned}\overrightarrow{P_1P_2} &= \langle 0-(-2), 2-1, 3-1\rangle \\ &= \langle 2,1,2\rangle, \\ \overrightarrow{P_1P_3} &= \langle 1-(-2), 0-1, -1-1\rangle \\ &= \langle 3,-1,-2\rangle.\end{aligned}$$

Then

$$\begin{aligned}\mathbf{n} &= \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} \\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 3 & -1 & -2\end{vmatrix} \\ &= (-2+2)\mathbf{i} - (-4-6)\mathbf{j} + (-2-3)\mathbf{k} \\ &= 10\mathbf{j} - 5\mathbf{k}.\end{aligned}$$

Hence an equation to the plane is

$$\begin{aligned} & 0(x-(-2)) + 10(y-1) + (-5)(z-1) = 0 \\ \Rightarrow \quad & 10y - 10 - 5z + 5 = 0 \\ \Rightarrow \quad & 10y - 5z - 5 = 0 \\ \Rightarrow \quad & 2y - z - 1 = 0\end{aligned}$$

Exercise: Use point on plane as P_2 and find the equation. (what is the resulting equation?) Do it with P_3 too. Also repeat with vectors $\overrightarrow{P_2P_3}$ and $\overrightarrow{P_2P_1}$ to find the normal. Write your conclusion.

(b) Left as exercise. ■

We now derive the general form of a plane

Theorem 4.1

If a, b, c, and d are constants, and a, b, and c are not all zero, then the graph of the equation

$$ax + by + cz + d = 0 \tag{35}$$

is a plane that has the vector $\mathbf{n} = \langle a, b, c\rangle$ as a normal.

Proof. Since a, b and c not all zero, there is at least one point (x_0, y_0, z_0) whose coordinates satisfy (35). For example if $a \neq 0$, then if we let $(x_0, y_0, z_0) = \left(-\frac{d}{a}, 0, 0\right)$ we see that

$$ax_0 + by_0 + cz_0 + d = a\left(-\frac{d}{a}\right) + b \times 0 + c \times 0 + d = -d + d = 0.$$

Similarly we can find at least point in the case of $b \neq 0, c \neq 0$. Hence we always get a point $P_0(x_0, y_0, z_0)$ on 3-space such that

$$ax_0 + by_0 + cz_0 + d = 0.$$

Subtracting it from the given equation in (35), we get

$$\begin{aligned}(ax + by + cz + d) - (ax_0 + by_0 + cz_0 + d) &= 0\\ \Rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0\end{aligned}$$

which represents a plane passing through $P_0(x_0, y_0, z_0)$ and having $\mathbf{n} = \langle a, b, c\rangle$ as the normal, by (34). ■

Problem 4.3

Determine whether the planes are parallel, perpendicular, or neither.

1. (a) $2x - 8y - 6z - 2 = 0, \quad -x + 4y + 3z - 5 = 0$
   (b) $3x - 2y + z = 1, \quad 4x + 5y - 2z = 4$
   (c) $x - y + 3z - 2 = 0, \quad 2x + z = 1$
2. (a) $3x - 2y + z = 4, \quad 6x - 4y + 3z = 7$
   (b) $y = 4x - 2z + 3, \quad x = \frac{1}{4}y + \frac{1}{2}z$
   (c) $x + 4y + 7z = 3, \quad 5x - 3y + z = 0$

Solution. Method : From the equation determine normal to each plane. The planes are parallel or perpendicular, whenever the normals are parallel or perpendicular respectively.
1(b) The normal vectors are

$$\mathbf{n}_1 = \langle 3, -2, 1\rangle \quad \text{and} \quad \mathbf{n}_2 = \langle 4, 5, -2\rangle$$

Clearly $\mathbf{n}_1$ and $\mathbf{n}_2$ are not parallel, so the planes are not parallel. Also

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = 3 \times 4 + (-2) \times 5 + 1 \times (-2) = 0$$

so the planes are perpendicular.
2(b) The equations can be rewritten as

$$4x - y - 2z + 3 = 0, \quad x - \frac{1}{4}y - \frac{1}{2}z = 0.$$

So the normal vectors are

$$\mathbf{n}_1 = \langle 4, -1, -2\rangle \quad \text{and} \quad \mathbf{n}_2 = \left\langle 1, -\frac{1}{4}, -\frac{1}{2}\right\rangle.$$

Clearly

$$\mathbf{n}_1 = 4\mathbf{n}_2$$

so that the planes are parallel.

Other problems are left as exercise. ■

Problem 4.4

Determine whether the line and plane are parallel, perpendicular, or neither.

1. (a) $x = 4 + 2t, \quad y = -t, \quad z = -1 - 4t;$
   $3x + 2y + z - 7 = 0$

   (b) $x = t, \quad y = 2t, \quad z = 3t;$
   $x - y + 2z = 5$

   (c) $x = -1 + 2t, \quad y = 4 + t, \quad z = 1 - t;$
   $4x + 2y - 2z = 7$

2. (a) $x = 3 - t, \quad y = 2 + t, \quad z = 1 - 3t;$
   $2x + 2y - 5 = 0$

   (b) $x = 1 - 2t, \quad y = t, \quad z = -t;$
   $6x - 3y + 3z = 1$

   (c) $x = t, \quad y = 1 - t, \quad z = 2 + t;$
   $x + y + z = 1$

Solution. Method : Find normal $\mathbf{n}$ to the plane and a vector $\mathbf{v}$ parallel to the line. If $\mathbf{n}$ is parallel to $\mathbf{v}$, then the line is perpendicular to the plane. If $\mathbf{n}$ is perpendicular to $\mathbf{v}$, then the line is parallel to the plane.

1(a) Here normal $\mathbf{n}$ to the plane is

$$\mathbf{n} = \langle 3, 2, 1 \rangle$$

and the vector $\mathbf{v}$ parallel to the line is

$$\mathbf{v} = \langle 2, -1, -4 \rangle .$$

Since $\mathbf{n}$ and $\mathbf{v}$ are not parallel, the line is not perpendicular to the plane.

Also

$$\mathbf{n} \cdot \mathbf{v} = 3 \times 2 + 2 \times (-1) + 1 \times (-4) = 6 - 2 - 4 = 0$$

so that $\mathbf{n}$ is perpendicular to $\mathbf{v}$, which imply that the line is parallel to the plane.

1(b) Here normal $\mathbf{n}$ to the plane is

$$\mathbf{n} = \langle 1, -1, 2 \rangle$$

and the vector $\mathbf{v}$ parallel to the line is

$$\mathbf{v} = \langle 1, 2, 3 \rangle .$$

Clearly $\mathbf{n}$ is not parallel to $\mathbf{v}$. Also

$$\mathbf{n} \cdot \mathbf{v} = 1 \times 1 + (-1) \times 2 + 2 \times 3 = 1 - 2 + 6 = 5 \neq 0$$

so that $\mathbf{n}$ is not perpendicular to $\mathbf{v}$.

Hence the line is neither parallel nor perpendicular to the plane.

2(b) Here normal $\mathbf{n}$ to the plane is

$$\mathbf{n} = \langle 6, -3, 3 \rangle$$

and the vector $\mathbf{v}$ parallel to the line is

$$\mathbf{v} = \langle -2, 1, -1 \rangle .$$

Clearly $\mathbf{n} = -3\mathbf{v}$ so that $\mathbf{n}$ is parallel to $\mathbf{v}$. Hence the line is perpendicular to the plane.

▶ Other problems are left as exercise. ■

Problem 4.5

Determine whether the line and plane intersect; if so, find the coordinates of the intersection.

1. (a) $x = t, \quad y = t, \quad z = t;$
   $3x - 2y + z - 5 = 0$

   (b) $x = 2 - t, \quad y = 3 + t, \quad z = t;$
   $2x + y + z = 1$

2. (a) $x = 3t, \quad y = 5t, \quad z = -t;$
   $2x - y + z + 1 = 0$

   (b) $x = 1 + t, \quad y = -1 + 3t, \quad z = 2 + 4t;$
   $x - y + 4z = 7$

Solution. Method : Substitute the values of x, y and z in t from the line to the equation of plane and solve for t. If the line interprets the plane, we get a value for t and the point of intersection will be obtained from the equation of line. If they do not intersect, we will not get a value for t

1(a) Substituting $x = t, y = t$, and $z = t$ in the equation of plane, we get

$$3t - 2t + t - 5 = 0 \Rightarrow 2t = 5 \Rightarrow t = \frac{5}{2}.$$

Hence the line intersect the plane at $t = \frac{5}{2}$. The point of intersection is

$$\left(\frac{5}{2}, \frac{5}{2}, \frac{5}{2}\right)$$

1(b) Substituting $x = 2 - t, y = 3 + t$, and $z = t$ in the equation of plane, we get

$$2(2 - t) + 3 + t + t = 1 \Rightarrow 4 - 2t + 3 + 2t = 1 \Rightarrow 7 = 1.$$

Hence the line does not intersect the plane.
Problems 2(a), (b) are left as exercise. ■

INTERSECTING PLANES

Two distinct intersecting planes determine two positive angles of intersection - an (acute) angle θ that satisfies the condition $0 \leq \theta \leq \frac{\pi}{2}$ and the supplement of that angle. If $\mathbf{n}_1$ and $\mathbf{n}_2$ are normals to the planes, then depending on the directions of $\mathbf{n}_1$ and $\mathbf{n}_2$, the angle θ is either the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ or the angle between $\mathbf{n}_1$ and $-\mathbf{n}_2$. In both cases, from Theorem 2.4, the *acute angle θ between the planes* is given by:

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} \tag{36}$$

Proof. If θ is the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$, then by (10)

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}$$

and if θ is the angle between $\mathbf{n}_1$ and $-\mathbf{n}_2$, then

$$\cos\theta = -\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}.$$

That is

$$\cos\theta = \pm\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}.$$

Since $0 \leq \theta \leq \frac{\pi}{2}$, we have

$$\cos\theta \geq 0 \;\Rightarrow\; \pm\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} \geq 0$$

so that

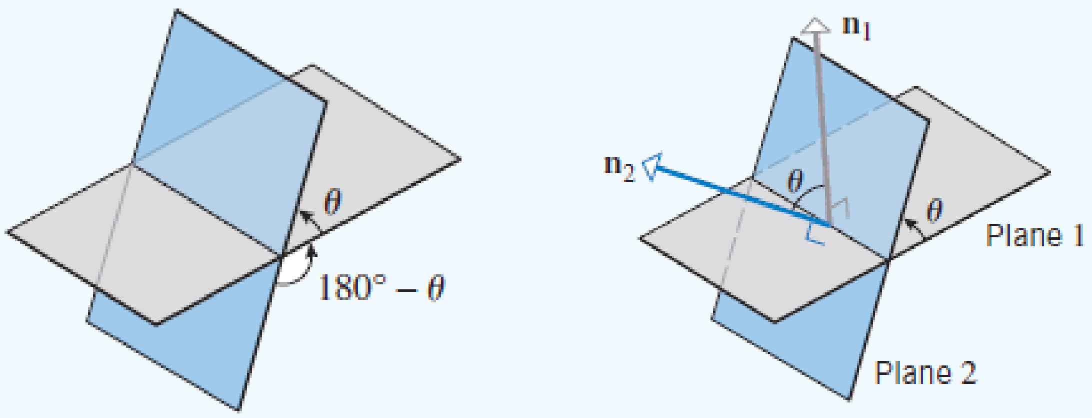
$$\pm\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \left|\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}\right| = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}.$$

Hence we get

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}.$$

■

Figure 4.3



Problem 4.6

Find the acute angle of intersection of the planes to the nearest degree.

(a) $x + 2y - 2z = 5$ and $6x - 3y + 2z = 8$

(b) $2x - 4y + 4z = 6$ and $6x + 2y - 3z = 4$

(c) $x = 0$ and $2x - y + z - 4 = 0$

Solution. (a) We have

$$\mathbf{n}_1 = \langle 1, 2, -2 \rangle \quad \text{and} \quad \mathbf{n}_2 = \langle 6, -3, 2 \rangle .$$

Then

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = 1 \times 6 + 2 \times (-3) + (-2) \times 2 = -4.$$

and

$$\|\mathbf{n}_1\| = \sqrt{9} = 3 \;\text{ and }\; \|\mathbf{n}_2\| = \sqrt{49} = 7.$$

Thus the acute angle θ between the planes is given by

$$\cos\theta = \frac{4}{3 \times 7} = \frac{4}{21}$$

which shows that

$$\theta \approx 79.02^\circ.$$

(b) Left as exercise.

(c) We have

$$\mathbf{n}_1 = \langle 1, 0, 0 \rangle \quad \text{and} \quad \mathbf{n}_2 = \langle 2, -1, 1 \rangle .$$

Then

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = 1 \times 2 + 0 \times (-1) + 0 \times 1 = 2.$$

and

$$\|\mathbf{n}_1\| = \sqrt{1} = 1 \quad \text{and} \quad \|\mathbf{n}_2\| = \sqrt{6}.$$

Thus the acute angle θ between the planes is given by

$$\cos\theta = \frac{2}{1 \times \sqrt{6}} = \frac{2}{\sqrt{6}}$$

which shows that

$$\theta \approx 35.26^\circ.$$

▶ It is clear that two planes intersect at a line.

Problem 4.7

Find parametric equations of the line of intersection of the planes.

(a) $-2x + 3y + 7z + 2 = 0$ and $x + 2y - 3z + 5 = 0$

(b) $3x - 5y + 2z = 0$ and $z = 0$.

(c) $2x - 4y + 4z = 6$ and $6x + 2y - 3z = 4$.

Solution. Method: Take L as line of intersection. We need a vector parallel to L and a point on L. Take normals $\mathbf{n}_1, \mathbf{n}_2$ of the planes. Since the line L is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$, it will be parallel to a vector orthogonal to both $\mathbf{n}_1$ and $\mathbf{n}_2$. We know that $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$ is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$. Hence we choose $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$ as the vector parallel to L. To find a point on L, we solve the equations of the plane, since any point on L lies on both of the planes.

(a) We have the normal vectors

$$\mathbf{n}_1 = \langle -2, 3, 7 \rangle \quad \text{and} \quad \mathbf{n}_2 = \langle 1, 2, -3 \rangle.$$

Suppose L be the line of intersection. The vector $\mathbf{v}$ parallel to L is given by

$$\begin{aligned}
\mathbf{v} &= \mathbf{n}_1 \times \mathbf{n}_2 \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 7 \\ 1 & 2 & -3 \end{vmatrix} \\
&= (-9 - 14)\mathbf{i} - (6 - 7)\mathbf{j} + (-4 - 3)\mathbf{k} \\
&= -23\mathbf{i} + \mathbf{j} - 7\mathbf{k}.
\end{aligned}$$

Let (x_0, y_0, z_0) be a point on L. Then (x_0, y_0, z_0) lies on both planes, so that

$$\begin{aligned}
-2x_0 + 3y_0 + 7z_0 + 2 &= 0 \\
x_0 + 2y_0 - 3z_0 + 5 &= 0
\end{aligned}$$

These are two equations in three variables. To solve, we find a value of one variable by identifying the nature of the line. Since the line is parallel to $\mathbf{v} = \langle -23, 1, -7 \rangle$ and since

$$\mathbf{v} \cdot \mathbf{k} = \langle -23, 1, -7 \rangle \cdot \langle 0, 0, 1 \rangle = -7 \neq 0$$

the line is not perpendicular to the $z-$axis, so that it is not parallel to the xy-plane. Hence the line L intersect the xy-plane. In other words, we can set

$$z_0 = 0$$

which gives

$$\begin{aligned} -2x_0 + 3y_0 &= -2 \\ x_0 + 2y_0 &= -5. \end{aligned}$$

Solving these we get $x_0 = -\frac{11}{7}, y_0 = -\frac{12}{7}$. Hence

$$\left(-\frac{11}{7}, -\frac{12}{7}, 0 \right)$$

is a point on L. Thus the parametric equations of the line are

$$x = -\frac{11}{7} - 23t, \quad y = -\frac{12}{7} + t, \quad z = 0 - 7t = -7t.$$

(note that you can see that L intersect xz and yz planes too.)
(b) and (c) are left as exercise. ■

Problem 4.8

Find an equation of the plane that satisfies the stated conditions.

1. The plane through the origin that is parallel to the plane $4x-2y+7z+12=0$.
2. The plane that contains the line $x=-2+3t, y=4+2t, z=3-t$ and is perpendicular to the plane $x-2y+z=5$.
3. The plane through the point $(-1,4,2)$ that contains the line of intersection of the planes $4x-y+z-2=0$ and $2x+y-2z-3=0$.
4. The plane through $(-1,4,-3)$ that is perpendicular to the line $x-2=t, y+3=2t, z=-t$.
5. The plane through $(1,2,-1)$ that is perpendicular to the line of intersection of the planes $2x+y+z=2$ and $x+2y+z=3$.
6. The plane through the points $P_1(-2,1,4), P_2(1,0,3)$ that is perpendicular to the plane $4x-y+3z=2$.
7. The plane through $(-1,2,-5)$ that is perpendicular to the planes $2x-y+z=1$ and $x+y-2z=3$.
8. The plane that contains the point $(2,0,3)$ and the line $x=-1+t, y=t, z=-4+2t$.
9. The plane whose points are equidistant from $(2,-1,1)$ and $(3,1,5)$.
10. The plane that contains the line $x=3t, y=1+t, z=2t$ and is parallel to the intersection of the planes $y+z=-1$ and $2x-y+z=0$.

Solution. 1. Let P be the required plane. The vector $\mathbf{n}=\langle 4,-2,7\rangle$ is a normal to the given plane. Since the P is parallel to the given plane,

$$\mathbf{n}=\langle 4,-2,7\rangle$$

is a normal vector to P. Since it passes through the origin, equation of the plane P is

$$4(x-0)+(-2)(y-0)+7(z-0)=0 \Rightarrow 4x-2y+7z=0.$$

2. Let P be the required plane. Since it contains the line $x=-2+3t, y=4+2t, z=3-t$ and the line passes through $P_0(-2,4,3)$, it is a point on P.

   The line is parallel to the vector $\mathbf{v}=\langle 3,2,-1\rangle$ and the given plane is perpendicular to the vector $\mathbf{u}=\langle 1,-2,1\rangle$. Hence the plane P is parallel to the vectors $\mathbf{v},\mathbf{u}$. Thus a normal to the plane P is given by

$$\mathbf{n} = \mathbf{v} \times \mathbf{u} = \langle 0, -4, -8 \rangle . \quad \text{(calculations left as exercise)}$$

Hence the equation of P is

$$\begin{aligned} 0(x-(-2)) + (-4)(y-4) + (-8)(z-3) &= 0 \\ \Rightarrow -4y - 8z + 40 &= 0 \\ \Rightarrow y + 2z &= 10. \end{aligned}$$

3. Let P be the required plane. Suppose L be the line of intersection of the planes $4x - y + z - 2 = 0$ and $2x + y - 2z - 3 = 0$.

   Given that $P_1(-1, 4, 2)$ is a point on the plane. We find a normal $\mathbf{n}$ to P. For determining normal we first find two points on the line L and so on P by solving the equations of the given planes. We have

$$\begin{aligned} 4x - y + z - 2 &= 0 \\ 2x + y - 2z - 3 &= 0 \end{aligned}$$

   Adding these two we get

$$6x - z = 5$$

   which is a line on xz-plane. When $x = 0$, we get $z = -5$ and when $z = 1$, we get $6x = 6 \Rightarrow x = 1$. Hence

$$(x, z) = (0, -5), (1, 1)$$

   are points on this line.

   When $(x, z) = (0, -5)$, from the equation of second plane we have

$$0 + y + 10 - 3 = 0 \Rightarrow y = -7$$

   and when $(x, z) = (1, 1)$, again from the equation of second plane we have

$$2 + y - 2 - 3 = 0 \Rightarrow y = 3.$$

   Hence

$$P_2(0, -7, -5) \quad \text{and} \quad P_3(1, 3, 1)$$

   are points on L and hence on P. Thus,

$$P_1(-1, 4, 2), P_2(0, -7, -5), P_3(1, 3, 1)$$

   are points on P. Then $\overrightarrow{P_1P_2} = \langle 1, -11, -7 \rangle$ and $\overrightarrow{P_1P_3} = \langle 2, -1, -1 \rangle$ are parallel to P so that

$$\mathbf{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \langle 4, -13, 21 \rangle$$

is normal to P. Hence an equation of P is

$$4(x+1) - 13(y-4) + 21(z-2) = 0 \Rightarrow 4x - 12y + 21z = -14.$$

(Here we use P_1 as a point on P).

4. Let P be the required plane. The given line is parallel to the vector $\mathbf{n} = \langle 1, 2, -1 \rangle$. Since P is perpendicular to the given line, $\mathbf{n}$ is a normal to P. Also P contains the point $(-1, 4, -3)$. Hence equation of P is

$$1(x - (-1)) + 2(y-4) + (-1)(z - (-3)) = 0 \Rightarrow x + 2y - z + 1 - 8 - 3 = 0.$$

That is

$$x + 2y - z = 10.$$

5. Let P be the required plane. Let L be the line of intersection of the planes $2x+y+z = 2$ and $x + 2y + z = 3$. Since these planes are perpendicular to

$$\mathbf{n}_1 = \langle 2, 1, 1 \rangle, \quad \text{and} \quad \mathbf{n}_2 = \langle 1, 2, 1 \rangle,$$

the line L is perpendicular to the vector

$$\mathbf{n} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle -1, -1, 3 \rangle.$$

Hence $\mathbf{n} = \langle -1, -1, 3 \rangle$ is a normal to P and since P passes through $(1, 2, -1)$, an equation P is

$$-(x-1) - (y-2) + 3(z+1) = 0 \Rightarrow -x - y + 3z + 6 = 0,$$

that is

$$x + y - 3z = 6.$$

6. Let P be the required plane. Since P is perpendicular to the plane $4x - y + 3z = 2$, the normal $\mathbf{u} = \langle 4, -1, 3 \rangle$ of this plane is parallel to P. Moreover

$$\mathbf{v} = \overrightarrow{P_1P_2} = \langle 3, -1, -1 \rangle$$

is also parallel to P, since P_1, P_2 are points on P. Hence a normal to P is

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \langle 4, 13, -1 \rangle.$$

Also $P_1(-2, 1, 4)$ is a point on P. So an equation of P is given by

$$4(x+2) + 13(y-1) - (z-4) = 0 \Rightarrow 4x + 13y - z = 1$$

7. Let P be the required plane. Then $(-1,2,-5)$ is a point on P. Clearly $\overrightarrow{n}_1 = \langle 2,-1,1\rangle$ and $\overrightarrow{n}_2 = \langle 1,1,-2\rangle$ are normals to the planes $2x-y+z=1$ and $x+y-2z=3$. Since P is perpendicular to these planes $\overrightarrow{n}_1$ and $\overrightarrow{n}_2$ are parallel to P. Hence a normal to P is given by

$$\mathbf{n} = \overrightarrow{n}_1 \times \overrightarrow{n}_2 = \langle 1,5,3\rangle .$$

Hence an equation to P is given by

$$(x+1)+5(y-2)+3(z+5)=0 \Rightarrow x+5y+3z=-6.$$

8. Let P be the required plane. So $P_1(2,0,3)$ is a point on P. Since the given line is parallel to $\mathbf{u} = \langle 1,1,2\rangle$, and since P contains this line, $\mathbf{u}$ is parallel to P also.

Clearly $P_2(-1,0,-4)$ is on the line and so is on P. Hence

$$\mathbf{v} = \overrightarrow{P_1P_2} = \langle -3,0,-7\rangle$$

is also parallel to P. Hence a normal to P is given by

$$\mathbf{n} = \mathbf{u}\times\mathbf{v} = \langle -7,1,3\rangle$$

so that an equation of P is given by

$$-7(x-2)+(y-0)+3(z-3)=0 \Rightarrow -7x+y+3z+5=0$$

that is

$$7x-y-3z=5.$$

9. Let P be the required plane. Since the points of P are equidistant from $P_1(2,-1,1)$ and $P_2(3,1,5)$, the midpoint of the segment P_1P_2 is a point on P. That is

$$\left(\frac{5}{2},0,3\right)$$

is a point on P. (midpoint of P_1P_2 is given by $\dfrac{P_1+P_2}{2}$)

Also $\mathbf{n} = \overrightarrow{P_1P_2} = \langle 1,2,4\rangle$ is normal to P. Hence an equation of P is given by

$$1\left(x-\frac{5}{2}\right)+2(y-0)+4(z-3)=0 \Rightarrow x+2y+4z-\frac{29}{2}=0$$

that is

$$2x+4y+8z=29.$$

10. Let P be the required plane and let L be the given line. Clearly $(0,1,0)$ is a point on L, so that it is a point on P too. Now we need a normal to P.

Note that $\mathbf{u} = \langle 3, 1, 2 \rangle$ is parallel to L and so parallel to P. Now

$$\mathbf{n}_1 = \langle 0, 1, 1 \rangle \quad \text{and} \quad \mathbf{n}_2 = \langle 2, -1, 1 \rangle$$

are normals to the given planes $y+z = -1$ and $2x-y+z = 0$, so each is perpendicular to the line L' of intersection of these planes. Hence

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 2, 2, -2 \rangle$$

is parallel to L'. Since P is parallel to L' too, it is parallel to $\mathbf{v}$ also.

Thus normal to P is given by

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \langle -6, 10, 4 \rangle .$$

Hence an equation to P is

$$-6(x-0) + 10(y-1) + 4(z-0) = 0 \Rightarrow -6x + 10y + 4z - 10 = 0,$$

that is

$$-3x + 5y + 2z = 5.$$

■

Problem 4.9

1. Find parametric equations of the line through the point $(5, 0, -2)$ that is parallel to the planes $x - 4y + 2z = 0$ and $2x + 3y - z + 1 = 0$.

2. Let L be the line $x = 3t + 1, y = -5t, z = t$.

   (a) Show that L lies in the plane $2x + y - z = 2$.

   (b) Show that L is parallel to the plane $x + y + 2z = 0$. Is the line above, below, or on this plane?

3. Show that the lines

$$L1 : x = -2 + t, y = 3 + 2t, z = 4 - t$$
$$L2 : x = 3 - t, y = 4 - 2t, z = t$$

   are parallel and find an equation of the plane they determine.

4. Show that the lines

$$L1 : x + 1 = 4t, y - 3 = t, z - 1 = 0$$
$$L2 : x + 13 = 12t, y - 1 = 6t, z - 2 = 3t$$

   intersect and find an equation of the plane they determine.

Solution. 1. Let L be the required line. Then $(5,0,-2)$ is a point on it. The vectors

$$\mathbf{n}_1 = \langle 1,-4,2\rangle \quad \text{and} \quad \mathbf{n}_2 = \langle 2,3,-1\rangle$$

are normals to the given planes. Since L is parallel to these planes, $\mathbf{n}_1$ and $\mathbf{n}_2$ are perpendicular to L. So

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle -2,5,11\rangle$$

is parallel to L. Thus parametric equations of L are

$$x = 5-2t, \quad y = 5t, \quad z = -2+11t.$$

2. (a) It is clear that L lies in the given plane, since,

$$2(3t+1)-5t-t = 2.$$

(b) First part is left as exercise.

Second part. Consider the equation $x+y+2z=0$ of the plane and the equation $x=3t+1, y=-5t, z=t$ of L. Then

$$(3t+1)+(-5t)+2t = 1$$

so that the line L lies on the plane $x+y+2z=1$, which is parallel to $x+y+2z=0$ and lies above it. Hence the L lies above the plane $x+y+2z=0$.

3. Clearly the vectors

$$\mathbf{v}_1 = \langle 1,2,-1\rangle \quad \text{and} \quad \mathbf{v}_2 = \langle -1,-2,1\rangle$$

are parallel to L_1 and L_2 respectively. Since $\mathbf{v}_1 = -\mathbf{v}_2$, the lines are parallel.

Let P be the plane determined by the lines L_1 and L_2. Then $P_1(-2,3,4)$ and $P_2(3,4,0)$ are points on P (since they are points on L_1 and L_2 respectively). Hence P is parallel to the vector

$$\mathbf{u} = \overrightarrow{P_1P_2} = \langle 5,1,-4\rangle.$$

Moreover $\mathbf{v}_1$ is also parallel to P so that,

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{u} = \langle -7,-1,-9\rangle$$

is a normal to P. Hence equation to P is

$$-7(x+2)-(y-3)-9(z-4) = 0 \Rightarrow -7x-y-9z+25 = 0,$$

that is

$$7x+y+9z = 25.$$

(by choosing P_1 as a point on P).

4. Left as exercise.

   Hint: Choose the point of intersection as point on the plane and use the vectors parallel to the given lines to determine normal.

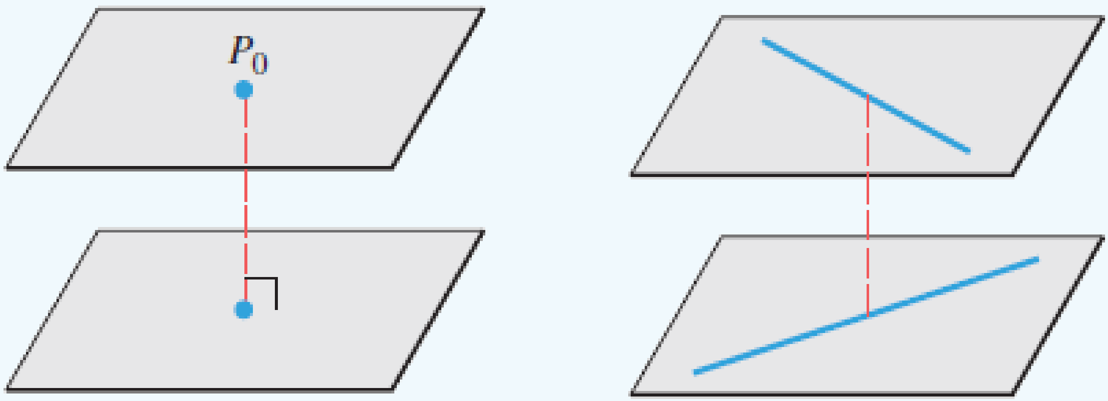
■

DISTANCE PROBLEMS INVOLVING PLANES

Next we will consider three basic distance problems in 3-space:

- Find the distance between a point and a plane.
- Find the distance between two parallel planes.
- Find the distance between two skew lines.

Figure 4.4



▶ The three problems are related. If we can find the distance between a point and a plane, then we can find the distance between parallel planes by computing the distance between one of the planes and an arbitrary point P_0 in the other plane.

▶ Moreover, we can find the distance between two skew lines by computing the distance between parallel planes containing them.

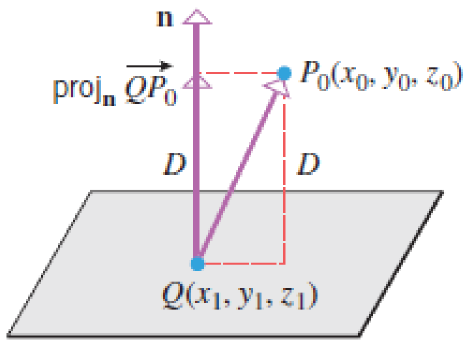
Theorem 4.2

The distance D between a point $P_0(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{37}$$

Proof.

Let $Q(x_1, y_1, z_1)$ be any point in the plane, and position the normal $\mathbf{n} = \langle a, b, c\rangle$ so that its initial point is at Q. From the figure it is clear that the distance D is equal to the length of the orthogonal projection of $\overrightarrow{QP_0}$ on $\mathbf{n}$.



Thus

$$D = \left\|\text{proj}_{\mathbf{n}}\overrightarrow{QP_0}\right\| = \left\|\left(\frac{\overrightarrow{QP_0}\cdot\mathbf{n}}{\|\mathbf{n}\|^2}\right)\mathbf{n}\right\| = \frac{|\overrightarrow{QP_0}\cdot\mathbf{n}|}{\|\mathbf{n}\|^2}\|\mathbf{n}\| = \frac{|\overrightarrow{QP_0}\cdot\mathbf{n}|}{\|\mathbf{n}\|}.$$

But

$$\begin{aligned}
\overrightarrow{QP_0} &= \langle x_0 - x_1, y_0 - y_1, z_0 - z_1\rangle \\
\Rightarrow \overrightarrow{QP_0}\cdot\mathbf{n} &= \langle x_0 - x_1, y_0 - y_1, z_0 - z_1\rangle\cdot\langle a, b, c\rangle \\
&= a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1) \\
&= (ax_0 + by_0 + cz_0) - (ax_1 + by_1 + cz_1)
\end{aligned}$$

Since (x_1, y_1, z_1) is a point on the plane we have

$$ax_1 + by_1 + cz_1 + d = 0 \Rightarrow ax_1 + by_1 + cz_1 = -d$$

so that

$$\overrightarrow{QP_0}\cdot\mathbf{n} = ax_0 + by_0 + cz_0 + d.$$

Also

$$\|\mathbf{n}\| = \sqrt{a^2 + b^2 + c^2}.$$

Hence

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

■

Problem 4.10

Show that a plane P is parallel to a plane $ax + by + cz + d = 0$ if and only if the equation of P is $ax + by + cz + d' = 0$ for some real number d'.

Solution. Suppose the equation of P is $ax + by + cz + d' = 0$. Then normal to P is $\mathbf{n} = \langle a, b, c\rangle$. Also normal to the given plane is $\langle a, b, c\rangle$, which shows that the planes are parallel.

Conversely assume that P is a plane parallel to $ax + by + cz + d = 0$. Suppose equation of P is

$$a_1x + b_1y + c_1z + d_1 = 0.$$

The normal to P is

$$\mathbf{n}_1 = \langle a_1, b_1, c_1 \rangle .$$

Since P is parallel to the plane $ax+by+cz+d=0$, the vector $\mathbf{n}_1$ is parallel to the normal $\mathbf{n} = \langle a, b, c \rangle$ so that $\mathbf{n}_1$ is a multiple of $\mathbf{n}$. Which shows that

$$\langle a_1, b_1, c_1 \rangle = k \langle a, b, c \rangle$$

for some $k \neq 0$ so that

$$a_1 = ka, \quad b_1 = kb, \quad c_1 = kc.$$

Hence equation of P becomes

$$\begin{aligned} kax + kby + kcz + d_1 &= 0 \\ \Rightarrow k(ax+by+cz) + d_1 &= 0 \\ \Rightarrow ax+by+cz+\frac{d_1}{k} &= 0 \\ \Rightarrow ax+by+cz+d' &= 0 \end{aligned}$$

where $d' = \frac{d_1}{k}$. ■

Problem 4.11

Show that the distance D between parallel planes

$$\begin{aligned} ax+by+cz+d_1 &= 0 \\ ax+by+cz+d_2 &= 0 \end{aligned}$$

is

$$\frac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}. \tag{38}$$

Solution. Let P_1, P_2 be the planes $ax+by+cz+d_1 = 0, ax+by+cz+d_2 = 0$ respectively. Since the planes are parallel, the distance between them is same as the distance of P_1 from a point of P_2.

Choose a point (x_0, y_0, z_0) on P_2, then

$$ax_0+by_0+cz_0+d_2 = 0 \Rightarrow ax_0+by_0+cz_0 = -d_2$$

and D is same as the distance of (x_0, y_0, z_0) from P_1, which is $ax+by+cz+d_1 = 0$. By (37),

$$D = \frac{|ax_0+by_0+cz_0+d_1|}{\sqrt{a^2+b^2+c^2}} = \frac{|-d_2+d_1|}{\sqrt{a^2+b^2+c^2}} = \frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}. \quad ■$$

Problem 4.12

1. Find the distance between the point and the plane.
   (a) $(1, -2, 3)$; $\quad 2x - 2y + z = 4$
   (b) $(0, 1, 5)$; $\quad 3x + 6y - 2z - 5 = 0$
2. Find the distance between the given parallel planes.
   (a) $-2x + y + z = 0, \quad 6x - 3y - 3z - 5 = 0$
   (b) $x + y + z = 1, \quad x + y + z = -1$

Solution. 1. (a) Equation of the given plane is

$$2x - 2y + z - 4 = 0.$$

Hence distance of $(1, -2, 3)$ from the plane is

$$d = \frac{|2 \times 1 - 2 \times (-2) + 3 - 4|}{\sqrt{2^2 + (-2)^2 + 1^2}} = \frac{5}{3}.$$

(b) Left as exercise.

2. (a) The given planes are

$$-2x + y + z = 0, \quad 6x - 3y - 3z - 5 = 0.$$

Rewriting the equation of second plane (divide by -3), we get

$$-2x + y + z = 0, \quad -2x + y + z + \frac{5}{3} = 0$$

so that the planes are parallel. Here $d_1 = 0, d_3 = \frac{5}{3}$ and $a = -2, b = 1, c = 1$. Hence distance between the planes is

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|0 - \frac{5}{3}|}{\sqrt{(-2)^2 + 1^2 + 1^2}} = \frac{5}{3\sqrt{6}}.$$

(b) Left as exercise.

■

Problem 4.13

Show that the lines

$$L_1 : x = 1 + 4t, \quad y = 5 - 4t, \quad z = -1 + 5t$$
$$L_2 : x = 2 + 8t, \quad y = 4 - 3t, \quad z = 5 + t$$

are skew and find the distance between them.

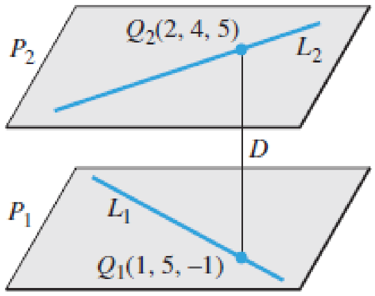
Solution. From problem 3.8, it is clear that the lines are skew.

As illustrated in figure 37, distance between L_1 and L_2 is same as the distance between the parallel planes containing them.

Method: Let P_1 and P_2 denote parallel planes containing L_1 and L_2, respectively. To find the distance D between L_1 and L_2, we will calculate the distance from a point in P_1 to the plane P_2. For this, we find a point on L_1 so that it is on P_1 and find the equation of P_2.

Substituting $t = 0$, we get

$$Q_1(1, 5, -1), Q_2(2, 4, 5)$$



as points on L_1 and L_2 respectively, that is on P_1 and P_2 respectively. Now we determine normal to P_2. Note that the vectors $\mathbf{u}_1 = \langle 4, -4, 5\rangle$ and $\mathbf{u}_2 = \langle 8, -3, 1\rangle$ are parallel to L_1 and L_2 respectively and hence parallel to P_1 and P_2 respectively. Since P_1 and P_2 are parallel planes, $\mathbf{u}_1$ and $\mathbf{u}_2$ are parallel to P_2. Hence a normal $\mathbf{n}$ to P_2 will be perpendicular to both $\mathbf{u}_1$ and $\mathbf{u}_2$, so that we choose

$$\begin{aligned} \mathbf{n} &= \mathbf{u}_1 \times \mathbf{u}_2 \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -4 & 5 \\ 8 & -3 & 1 \end{vmatrix} \\ &= 11\mathbf{i} + 36\mathbf{j} + 20\mathbf{k}. \end{aligned}$$

Since $Q_2(2, 4, 5)$ is a point on P_2, an equation of P_2 is given by

$$11(x-2) + 36(y-4) + 20(z-5) = 0 \Rightarrow 11x + 36y + 20z - 22 - 144 - 100 = 0$$

or

$$11x + 36y + 20z - 266 = 0.$$

Hence distance between P_1 and P_2, which is same as distance between $Q_1(1, 5, -1)$ and P_2 is given by

$$D = \frac{|11 \times 1 + 36 \times 5 + 20 \times (-1) - 266|}{\sqrt{11^2 + 36^2 + 20^2}} = \frac{95}{\sqrt{1817}}.$$

Thus distance between the lines L_1 and L_2 is also equal to $\dfrac{95}{\sqrt{1817}}$. ■

Problem 4.14

In each problem, show that the lines are skew and find the distance between them.

(a) $L_1 : x = 1 + 7t, \quad y = 3 + t, \quad z = 5 - 3t$
$L_2 : x = 4 - t, \quad y = 6, \quad z = 7 + 2t$

(b) $L_1 : x = 3 - t, \quad y = 4 + 4t, \quad z = 1 + 2t$
$L_2 : x = t, \quad y = 3, \quad z = 2t$

Solution. (a) Showing that the lines are skew is left as exercise. Let P_1 and P_2 denote parallel planes containing L_1 and L_2, respectively. Here we determine the distance between P_1 and a point on P_2.

Put $t = 0$ in L_1 and L_2, we get

$$Q_1(1, 3, 5), Q_2(4, 6, 7)$$

as points on L_1 and L_2, that is, on P_1 and P_2 respectively.

Since $\mathbf{u}_1 = \langle 7, 1, -3\rangle$ and $\mathbf{u}_2 = \langle -1, 0, 2\rangle$ are parallel to P_1, normal to P_1 is given by

$$\begin{aligned} \mathbf{n} &= \mathbf{u}_1 \times \mathbf{u}_2 \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 1 & -3 \\ -1 & 0 & 2 \end{vmatrix} \\ &= 2\mathbf{i} - 11\mathbf{j} + \mathbf{k}. \end{aligned}$$

Since P_1 passes through $Q_1(1, 3, 5)$, an equation to P_1 is

$$2(x - 1) - 11(y - 3) + 1(z - 5) = 0 \Rightarrow 2x - 11y + z + 26 = 0.$$

Hence distance between P_1 and P_2, which is same as distance between $Q_2(4, 6, 7)$ and P_1 is given by

$$D = \frac{|2 \times 4 - 11 \times 6 + 7 + 26|}{\sqrt{2^2 + (-11)^2 + 1^2}} = \frac{25}{\sqrt{126}}.$$

Thus distance between the lines L_1 and L_2 is also equal to $\dfrac{25}{\sqrt{126}}$.

(b) Left as exercise. ■

Problem 4.15

(a) Find an equation of the sphere with center $C(2,1,-3)$ that is tangent to the plane $P:\ x-3y+2z=4$.

(b) Locate the point of intersection of the plane $P:\ 2x+y-z=0$ and the line through $C(3,1,0)$ that is perpendicular to the plane.

(c) Show that the line $x=-1+t, y=3+2t, z=-t$ and the plane $2x-2y-2z+3=0$ are parallel, and find the distance between them.

Solution. (a) Left as exercise. (Hint: Radius of the sphere is same as the distance between C and P.)
(b) Left as exercise. (Hint: Vector parallel to the line is same as the vector normal to the plane P. (why?))
(c) Left as exercise. ■

5 QUADRIC SURFACES

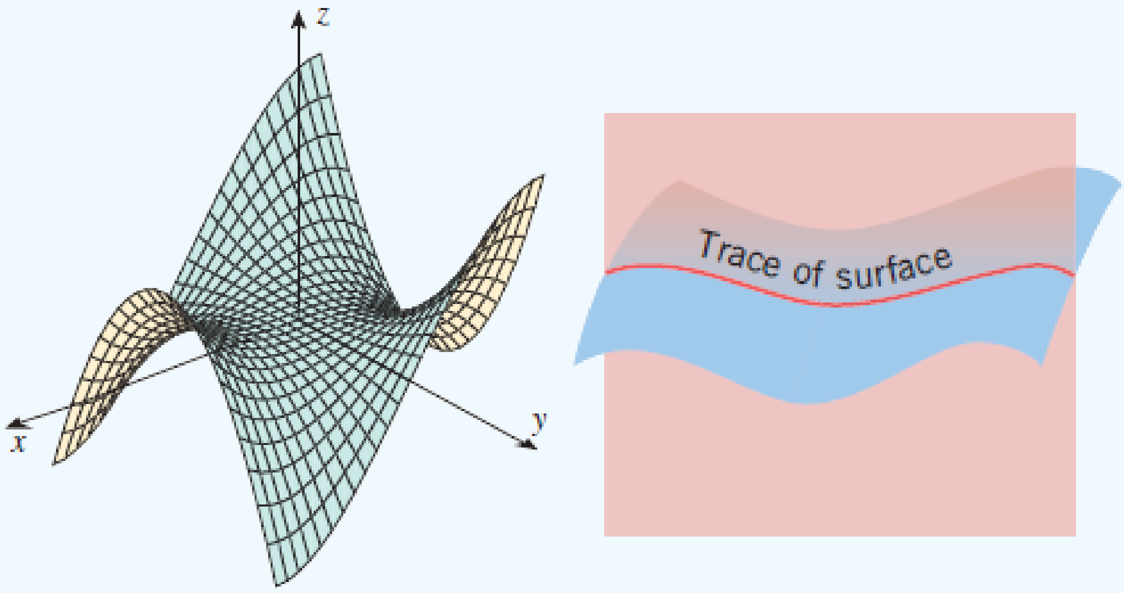
These are important class of surfaces that are the three-dimensional analogs of the conic sections.

TRACES OF SURFACES

Although the general shape of a curve in 2-space can be obtained by plotting points, this method is not usually helpful for surfaces in 3-space because too many points are required. It is more common to build up the shape of a surface with a network of ***mesh lines***, which are curves obtained by cutting the surface with well-chosen planes. For example, the figure below was generated by a CAS (Computer Algebra System), shows the graph of $z = x^3 - 3xy^2$ rendered with a combination of mesh lines and colorization to produce the surface detail. This surface is called a “monkey saddle” because a monkey sitting astride the surface has a place for its two legs and tail.

The mesh line that results when a surface is cut by a plane is called the ***trace*** of the surface in the plane.

Figure 5.1



A monkey saddle

For example, consider the surface

$$z = x^2 + y^2 \tag{39}$$

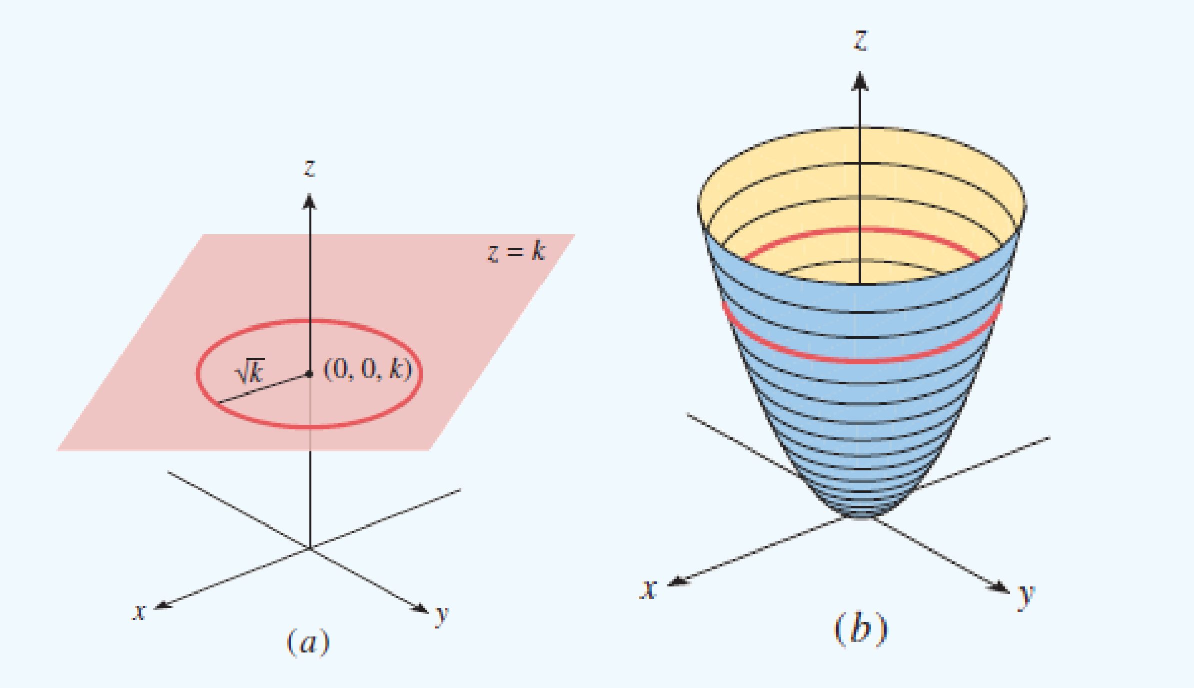
To find its trace in the plane $z = k$, we substitute this value of z into (39), which yields

$$x^2 + y^2 = k \quad (z = k) \tag{40}$$

- If $k < 0$, this equation has no real solutions, so there is no trace.
- However, if $k \geq 0$, then the graph of (40) is a circle of radius $\sqrt{k}$ centered at the point $(0, 0, k)$ on the z-axis. (see Figure 5.2)

Thus, for nonnegative values of k the traces parallel to the xy-plane form a family of circles, centered on the z-axis, whose radii start at zero and increase with k.

Figure 5.2



Now we examine the traces of (39) in planes parallel to the yz- plane. Such planes have equations of the form $x = k$, so we substitute this in (39) to obtain

$$z = y^2 + k^2 \quad (x = k) \tag{41}$$

- When $k = 0$ (41) becomes $z = y^2$, which is a parabola in the plane $x = 0$ (yz-plane) that has its vertex at the origin, opens in the positive z-direction, and is symmetric about the z-axis. (the blue parabola in Figure 5.3 a).
- For $k > 0$ the trace can be drawn as follows
  - We know that the graph of $z = y^2 + k^2$ is obtained by translating the parabola $z = y^2$, k^2 units in the positive direction of z-axis.
  - Since (41) is obtained when $x = k$, we draw the graph on the plane $x = k$, which is on the positive x-axis.

✏ Hence we get the parabola as in the case of $z = y^2$, but the new vertex will be at $(k, 0, k^2)$.

This is the red parabola in Figure 5.3 a.

▶ For $k < 0$ also we get the same parabola in the case of $k > 0$, but here in the negative x-axis.

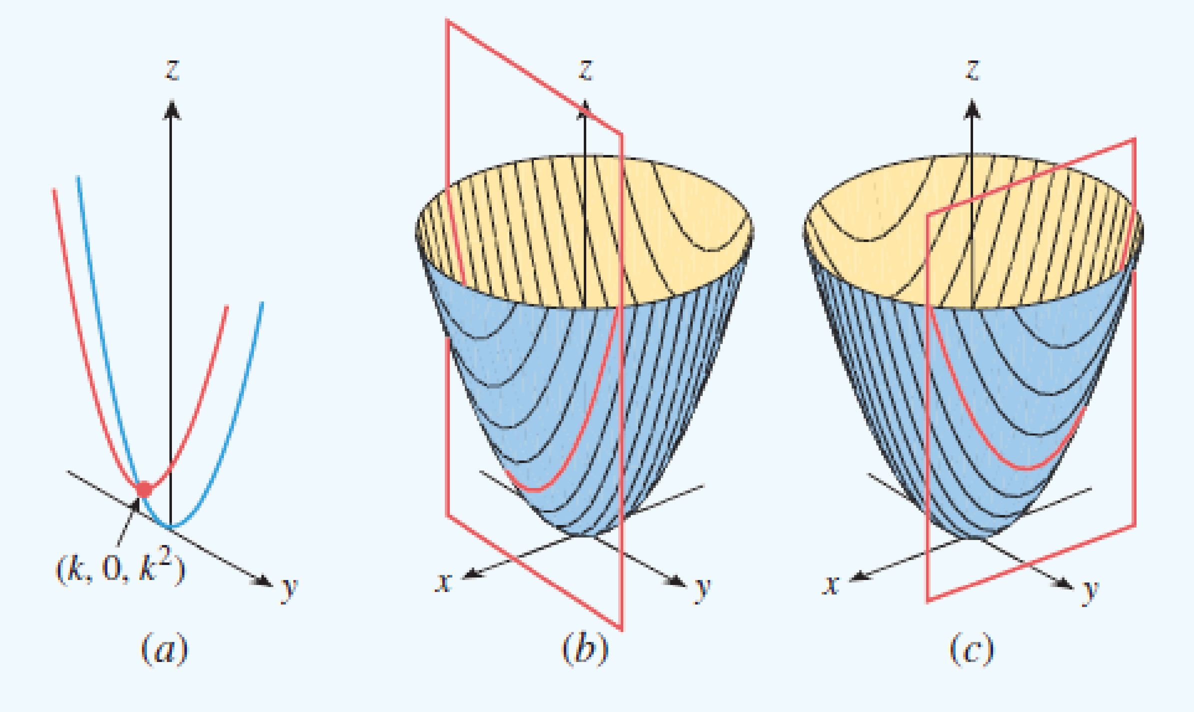
Thus, the traces in planes parallel to the yz-plane form a family of parabolas whose vertices move upward as k^2 increases (see Figure 5.3 b).

Similarly, the traces in planes parallel to the xz-plane have equations of the form

$$z = x^2 + k^2 \quad (y = k)$$

which again is a family of parabolas whose vertices move upward as k^2 increases Figure 5.3 c.

Figure 5.3



THE QUADRIC SURFACES

We have noted that a second-degree equation

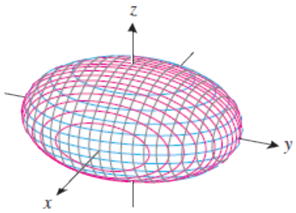
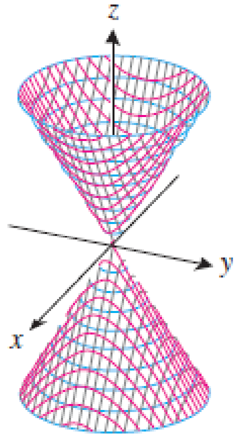
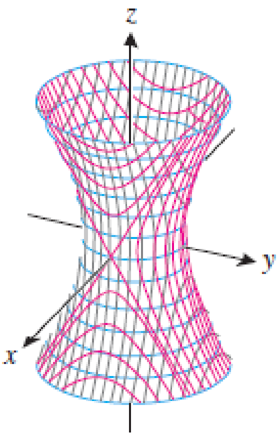
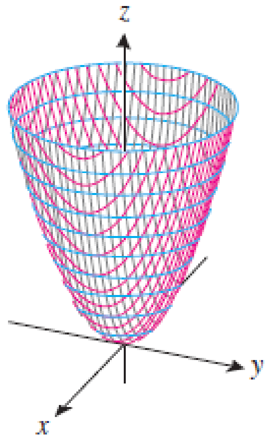
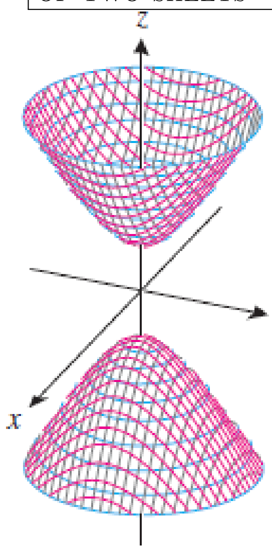
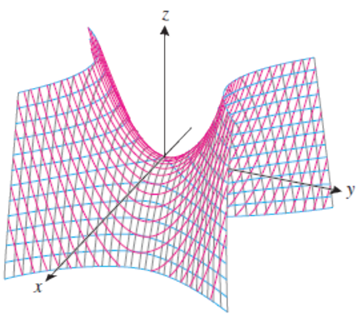
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

represents a conic section. The analog of this equation in an xyz coordinate system is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0 \tag{42}$$

which is called a ***second-degree equation*** in x, y, and z. The graphs of such equations are called ***quadric surfaces*** or sometimes ***quadrics***.

Six common types of quadric surfaces are shown below - *ellipsoids, hyperboloids of one sheet, hyperboloids of two sheets, elliptic cones, elliptic paraboloids*, and *hyperbolic paraboloids*. (The constants a, b, and c that appear in the equations in the table are assumed to be positive.)

SURFACE	EQUATION	SURFACE	EQUATION
ELLIPSOID 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ The traces in the coordinate planes are ellipses, as are the traces in those planes that are parallel to the coordinate planes and intersect the surface in more than one point.	ELLIPTIC CONE 	$z^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ The trace in the xy-plane is a point (the origin), and the traces in planes parallel to the xy-plane are ellipses. The traces in the yz and xz-planes are pairs of lines intersecting at the origin. The traces in planes parallel to these are hyperbolas.
HYPERBOLOID OF ONE SHEET 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ The trace in the xy-plane is an ellipse, as are the traces in planes parallel to the xy-plane. The traces in the yz-plane and xz-plane are hyperbolas, as are the traces in those planes that are parallel to these and do not pass through the x- or y-intercepts. At these intercepts the traces are pairs of intersecting lines.	ELLIPTIC PARABOLOID 	$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ The trace in the xy-plane is a point (the origin), and the traces in planes parallel to and above the xy-plane are ellipses. The traces in the yz- and xz-planes are parabolas, as are the traces in planes parallel to these.
HYPERBOLOID OF TWO SHEETS 	$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ There is no trace in the xy-plane. In planes parallel to the xy-plane that intersect the surface in more than one point the traces are ellipses. In the yz- and xz-planes, the traces are hyperbolas, as are the traces in those planes that are parallel to these.	HYPERBOLIC PARABOLOID 	$z=\frac{y^2}{b^2}-\frac{x^2}{a^2}$ The trace in the xy-plane is a pair of lines intersecting at the origin. The traces in planes parallel to the xy-plane are hyperbolas. The hyperbolas above the xy-plane open in the y-direction, and those below in the x-direction. The traces in the yz- and xz-planes are parabolas, as are the traces in planes parallel to these.

TECHNIQUES FOR GRAPHING QUADRIC SURFACES

A rough sketch of an ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad (a>0, b>0, c>0)$$

can be obtained by first plotting the intersections with the coordinate axes, and then sketching the elliptical traces in the coordinate planes.

Problem 5.1

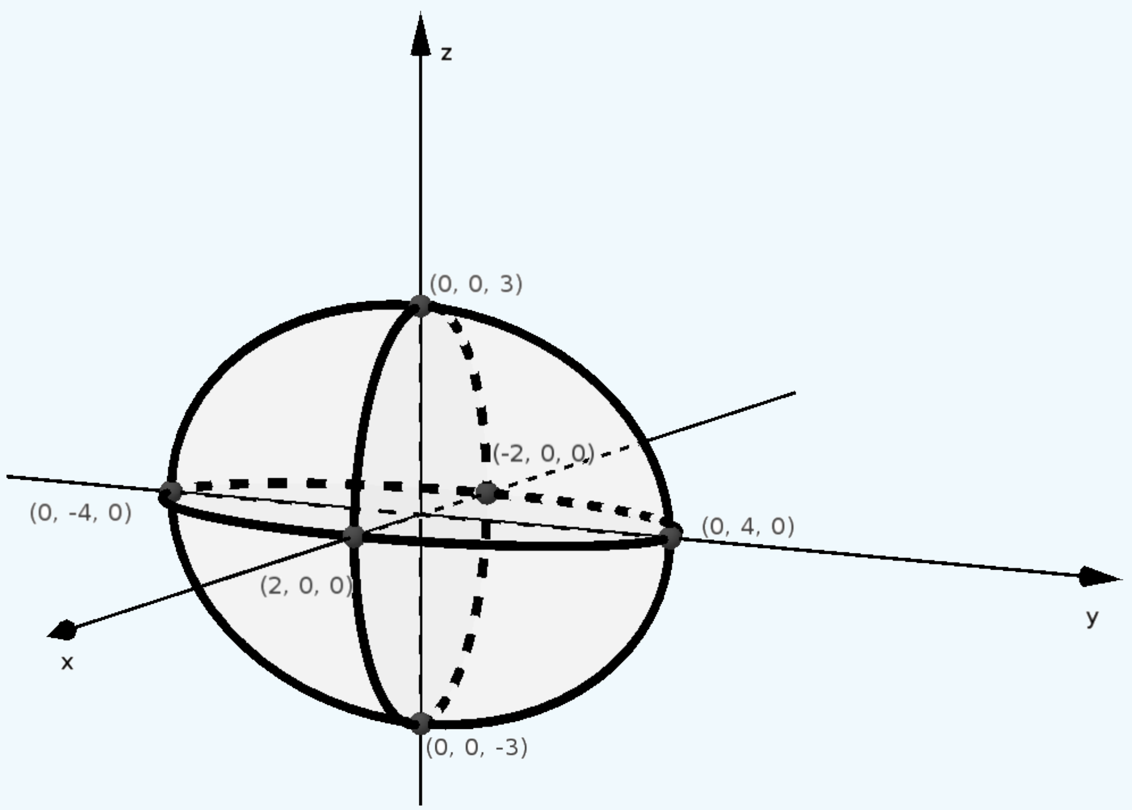
Sketch the ellipsoid

$$\frac{x^2}{4}+\frac{y^2}{16}+\frac{z^2}{9}=1$$

Solution. We first find the intercepts on coordinate axes. When $y=z=0$, we have $x^2=4 \Rightarrow x=\pm 2$. Hence x-intercepts are $(2,0,0)$ and $(-2,0,0)$. Similarly we get the y-intercepts $(0,\pm 4,0)$ and z-intercepts $(0,0,\pm 3)$.

Now we find traces in coordinate planes. In xy plane, we have $z=0$ so that we get the ellipse $\frac{x^2}{4}+\frac{y^2}{16}=1$. Similarly trace by xz-plane is the ellipse $\frac{x^2}{4}+\frac{z^2}{9}=1$ and trace by the yz-plane is the ellipse $\frac{y^2}{16}+\frac{z^2}{9}=1$.

Figure 5.4



Problem 5.2

Identify and sketch the surface

$$6x^2 + 3y^2 + 4z^2 = 12$$

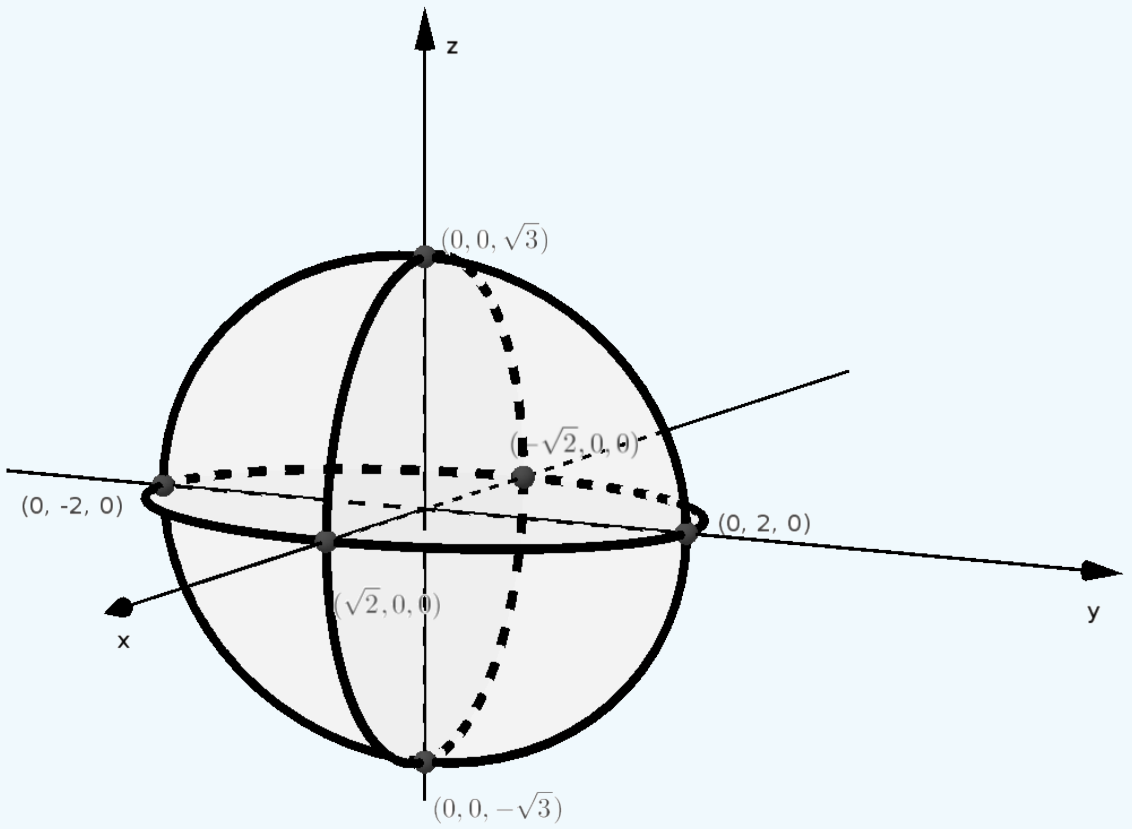
Solution. The given equation can be rewritten as

$$\frac{x^2}{2} + \frac{y^2}{4} + \frac{z^2}{3} = 1,$$

and the equation represents an ellipsoid. We first find the intercepts on coordinate axes. When $y = z = 0$, we have $x^2 = 2 \Rightarrow x = \pm\sqrt{2}$. Hence x-intercepts are $(\sqrt{2}, 0, 0)$ and $(-\sqrt{2}, 0, 0)$. Similarly we get the y-intercepts $(0, \pm 2, 0)$ and z-intercepts $(0, 0, \pm\sqrt{3})$.

Now we find traces in coordinate planes. In xy plane, we have $z = 0$ so that we get the ellipse $\frac{x^2}{2} + \frac{y^2}{4} = 1$. Similarly trace by xz-plane is the ellipse $\frac{x^2}{2} + \frac{z^2}{3} = 1$ and trace by the yz-plane is the ellipse $\frac{y^2}{4} + \frac{z^2}{3} = 1$.

Figure 5.5



■

A rough sketch of a hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad (a > 0, b > 0, c > 0)$$

can be obtained by first sketching the elliptical trace in the xy-plane, then the elliptical traces in the planes $z = \pm c$, and then the hyperbolic curves that join the endpoints of the axes of these ellipses.

Problem 5.3

Sketch the graph of the hyperboloid of one sheet

$$x^2 + y^2 - \frac{z^2}{4} = 1$$

Solution. The trace in the xy-plane, obtained by setting $z = 0$, is

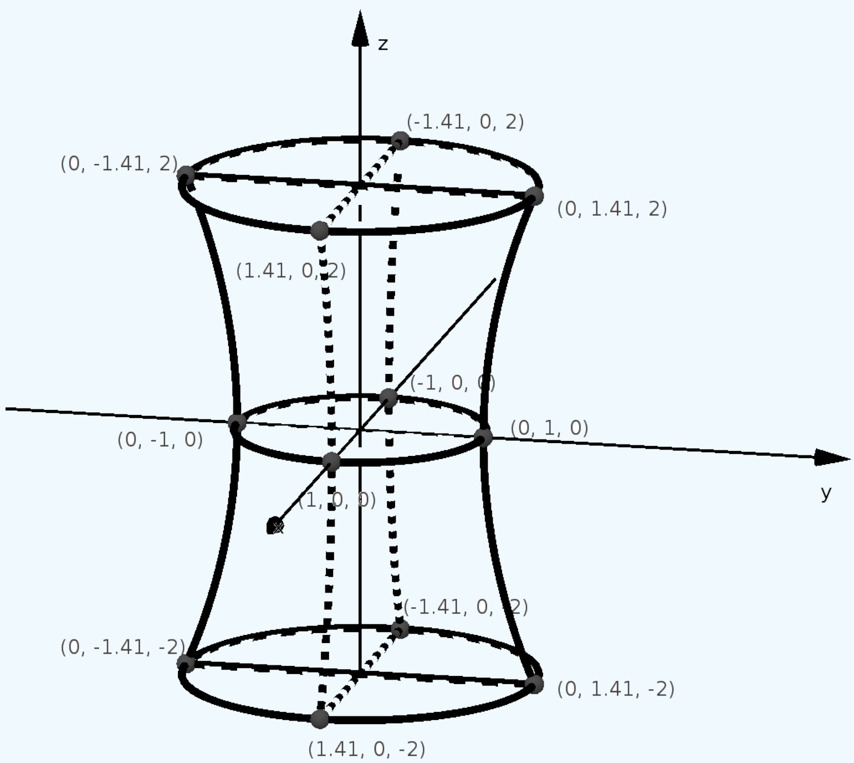
$$x^2 + y^2 = 1 \quad (z = 0)$$

which is a circle of radius 1 centered on the z-axis (origin). The traces in the planes $z = 2$ and $z = -2$, obtained by setting $z = \pm 2$, are given by

$$x^2 + y^2 = 2 \quad (z = \pm 2)$$

which are circles of radius $\sqrt{2}$ centered on the z-axis. Joining these circles by the hyperbolic traces in the vertical coordinate planes yields the graph.

Figure 5.6



■

Problem 5.4

Sketch the graph of the hyperboloid of one sheet

$$\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1$$

Solution. The trace in the xy-plane, obtained by setting $z = 0$, is

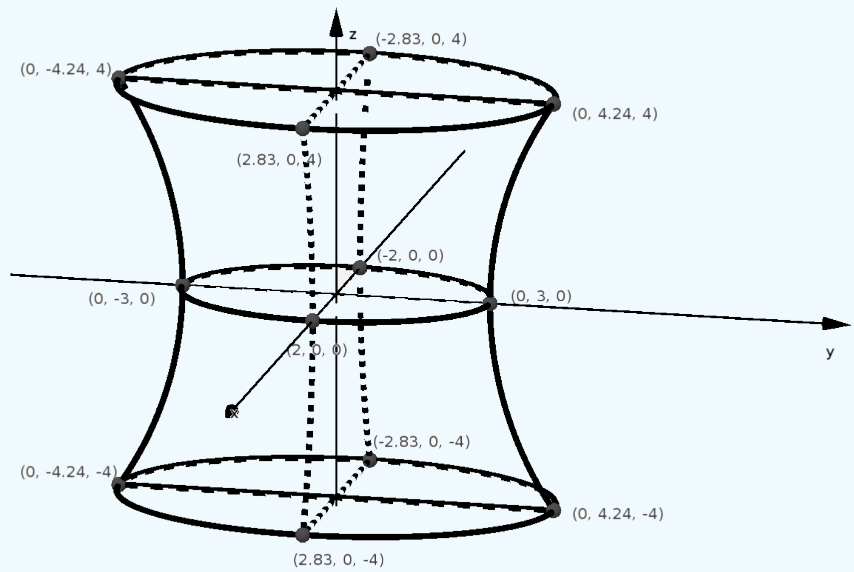
$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \quad (z = 0)$$

which is an ellipse centered on the z-axis (origin) and having major axis along y-axis $(a = 3, b = 2)$. The traces in the planes $z = 4$ and $z = -4$, obtained by setting $z = \pm 4$, are given by

$$\frac{x^2}{8} + \frac{y^2}{18} = 1 \quad (z = \pm 4)$$

which are ellipses centered on the z-axis with major axis along y-axis and $a = 3\sqrt{2} \approx 4.24, b = 2\sqrt{2} \approx 2.83$. Joining these ellipses by the hyperbolic traces in the vertical coordinate planes yields the graph.

Figure 5.7



■

A rough sketch of the hyperboloid of two sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (a > 0, b > 0, c > 0)$$

can be obtained by first plotting the intersections with the z-axis, then sketching the elliptical traces in the planes $z = \pm 2c$, and then sketching the hyperbolic traces that connect the z-axis intersections and the endpoints of the axes of the ellipses. (It is not essential to use the planes $z = \pm 2c$, but these are good choices since they simplify the calculations slightly and have the right spacing for a good sketch.)

Problem 5.5

Sketch the graph of the hyperboloid of two sheets

$$z^2 - x^2 - \frac{y^2}{4} = 1.$$

Solution. When $x = y = 0$ we get $z^2 = 1 \Rightarrow z = \pm 1$, so that the surface intersect the z axis at $(0, 0, 1)$ and $(0, 0, -1)$. Setting $z = \pm 2$, we get

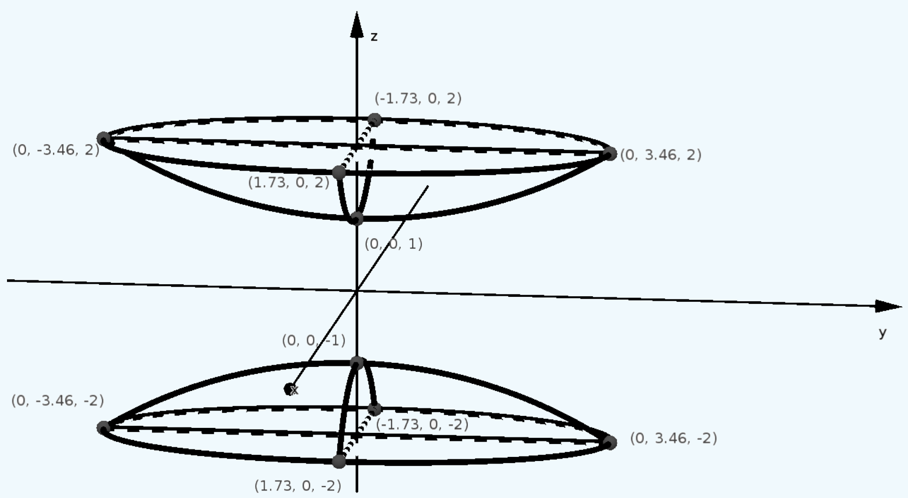
$$4 - x^2 - \frac{y^2}{4} = 1 \Rightarrow x^2 + \frac{y^2}{4} = 3 \Rightarrow \frac{x^2}{3} + \frac{y^2}{12} = 1.$$

That is the traces in the planes $z = 2$ and $z = -2$, obtained by

$$\frac{x^2}{3} + \frac{y^2}{12} = 1 \quad (z = \pm 2),$$

these are ellipses with centre on the planes $z = \pm 2$, and major axis parallel to y-axis and $a = \sqrt{12} \approx 3.46, b = \sqrt{3} =\approx 1.73$. Sketching these ellipses and the hyperbolic traces in the vertical coordinate planes yields,

Figure 5.8



■

Problem 5.6

Identify and sketch the surface

$$9z^2 - 4y^2 - 9x^2 = 36$$

Solution. The given equation can be written as

$$\frac{z^2}{4} - \frac{y^2}{9} - \frac{x^2}{4} = 1.$$

The surface is a hyperboloid of two sheets. When $x = y = 0$ we get $z^2 = 4 \Rightarrow z = \pm 2$, so that the surface intersect the z axis at $(0,0,2)$ and $(0,0,-2)$. Setting $z = \pm 4$, we get

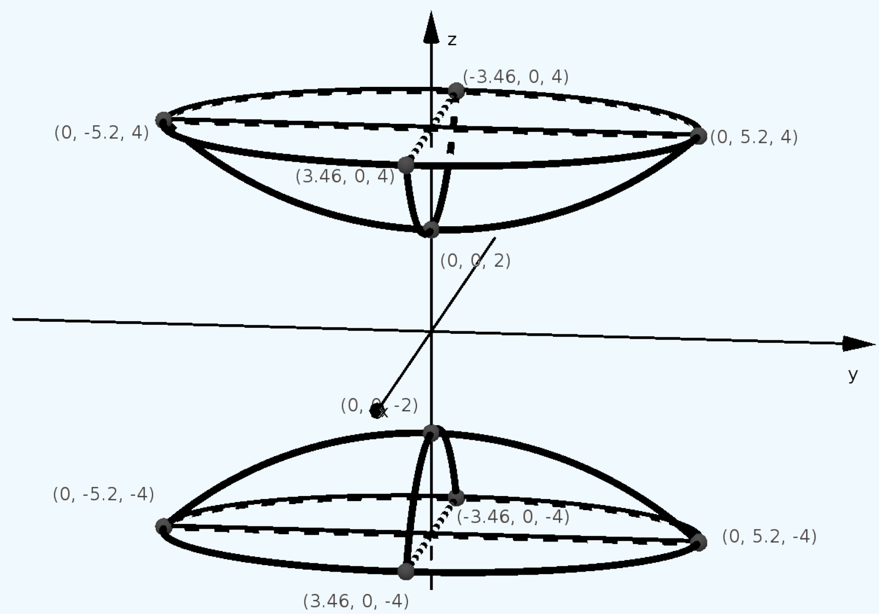
$$4 - \frac{y^2}{9} - \frac{x^2}{4} = 1 \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 3 \Rightarrow \frac{x^2}{12} + \frac{y^2}{27} = 1.$$

That is the traces in the planes $z = 4$ and $z = -4$, obtained by

$$\frac{x^2}{12} + \frac{y^2}{27} = 1 \quad (z = \pm 4),$$

these are ellipses with centre on the planes $z = \pm 4$, and major axis parallel to y-axis and $a = \sqrt{27} \approx 5.2, b = \sqrt{12} = \approx 3.46$. Sketching these ellipses and the hyperbolic traces in the vertical coordinate planes yields,

Figure 5.9



■

A rough sketch of the elliptic cone

$$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad (a > 0, b > 0)$$

can be obtained by first sketching the elliptical traces in the planes $z = \pm 1$ and then sketching the linear traces that connect the endpoints of the axes of the ellipses.

Problem 5.7

Sketch the graph of the elliptic cone

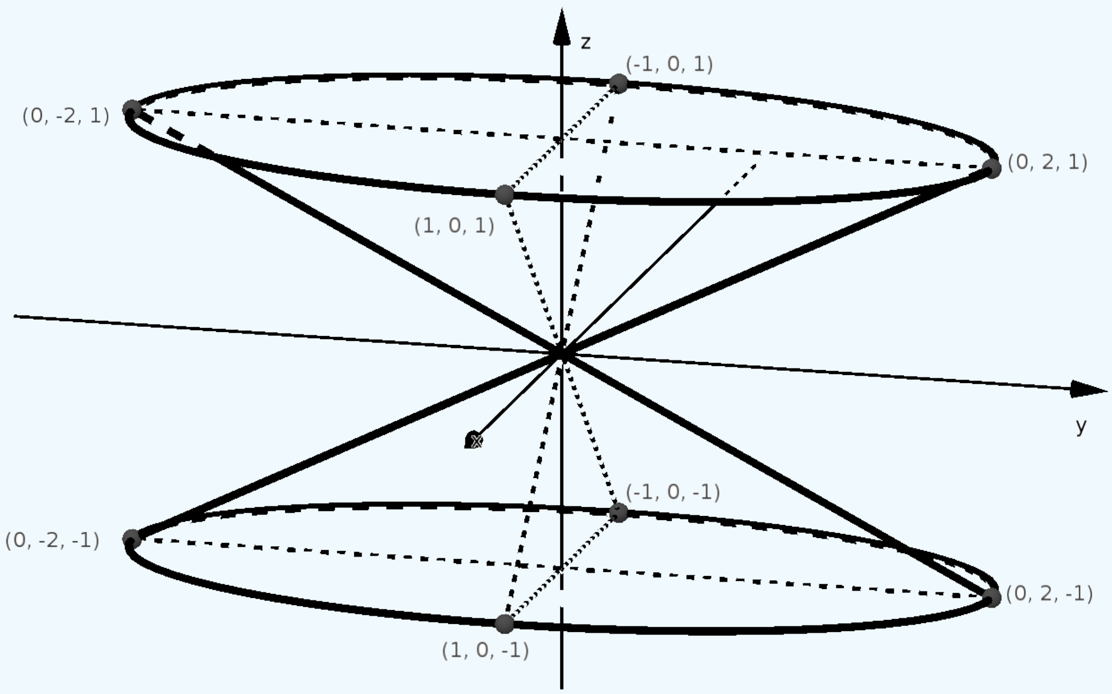
$$z^2 = x^2 + \frac{y^2}{4}.$$

Solution. When $z = \pm 1$, we get

$$x^2 + \frac{y^2}{4} = 1, \quad (z = \pm 1)$$

are ellipses with centres on the planes $z = \pm 1$ and major axis parallel to y-axis and $a = 2, b = 1$. Sketching these ellipses and the linear traces in the vertical coordinate planes yields the graph

Figure 5.10



Problem 5.8

Identify and sketch the surface

$$9x^2 + 4y^2 - 36z^2 = 0.$$

Solution. The given equation can be written as

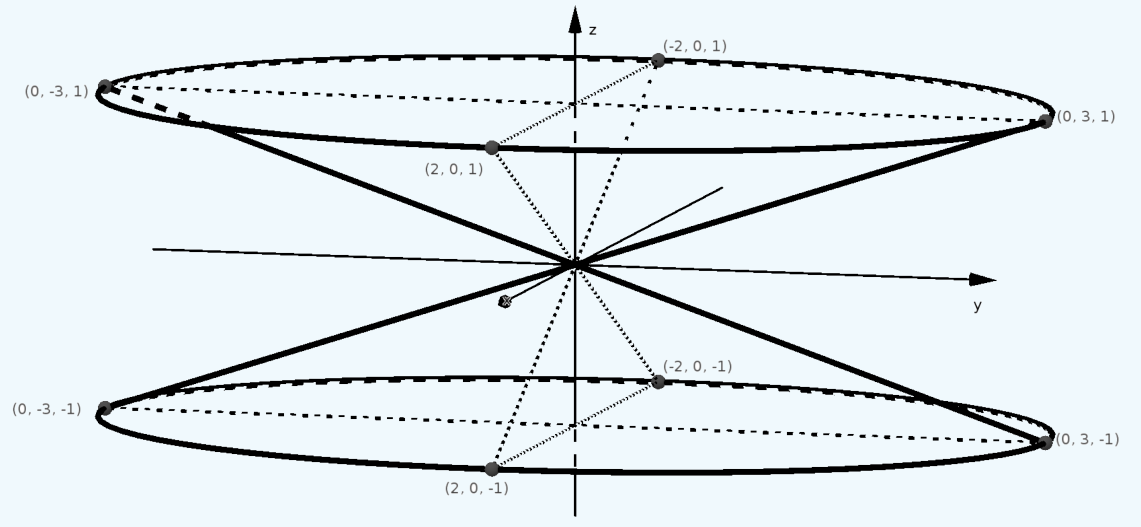
$$z^2 = \frac{x^2}{4} + \frac{y^2}{9}$$

which is an elliptic cone. When $z = \pm 1$, we get

$$\frac{x^2}{4} + \frac{y^2}{9} = 1, \quad (z = \pm 1)$$

are ellipses with centres on the planes $z = \pm 1$ and major axis parallel to y-axis and $a = 3, b = 2$. Sketching these ellipses and the linear traces in the vertical coordinate planes yields the graph

Figure 5.11



A rough sketch of the elliptic paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad (a > 0, b > 0)$$

can be obtained by first sketching the elliptical trace in the plane $z = 1$ and then sketching the parabolic traces in the vertical coordinate planes to connect the origin to the ends of the axes of the ellipse.

Problem 5.9

Sketch the graph of the elliptic paraboloid

$$z = \frac{x^2}{4} + \frac{y^2}{9}.$$

Solution. The trace in the plane $z = 1$ is

$$\frac{x^2}{4} + \frac{y^2}{9} = 1, \quad (z = 1)$$

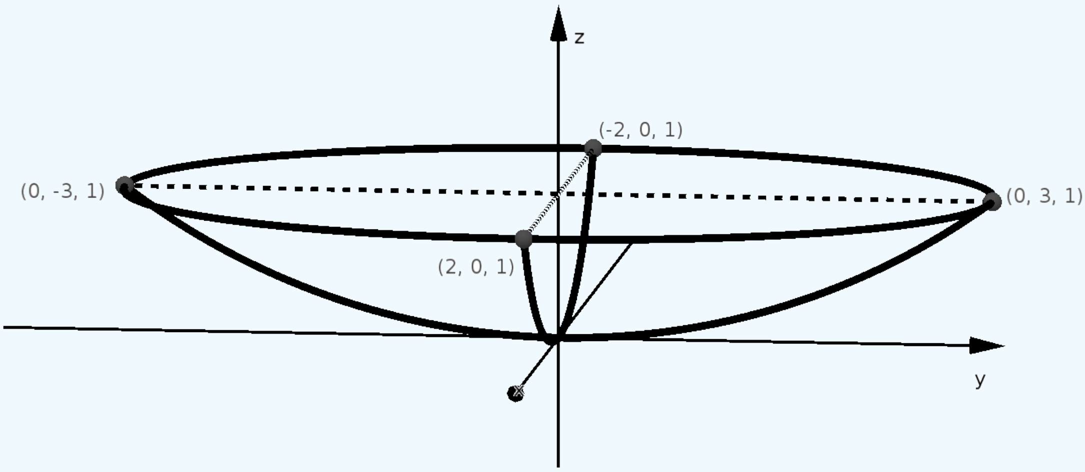
which is an ellipse with centre on the z-axis, major axis parallel to y-axis and $a = 3, b = 2$.

Traces in the xz (put $y = 0$) and yz (put $x = 0$) planes are

$$z = \frac{x^2}{4} \quad \text{and} \quad z = \frac{y^2}{9}.$$

They represent parabolas (as described in the case of (39)). Thus the graph is

Figure 5.12



■

Problem 5.10

Identify and sketch the surface

$$4z = x^2 + 2y^2.$$

Solution. The equation can be written as

$$z = \frac{x^2}{4} + \frac{y^2}{2}.$$

The trace in the plane $z = 1$ is

$$\frac{x^2}{4} + \frac{y^2}{2} = 1, \quad (z = 1)$$

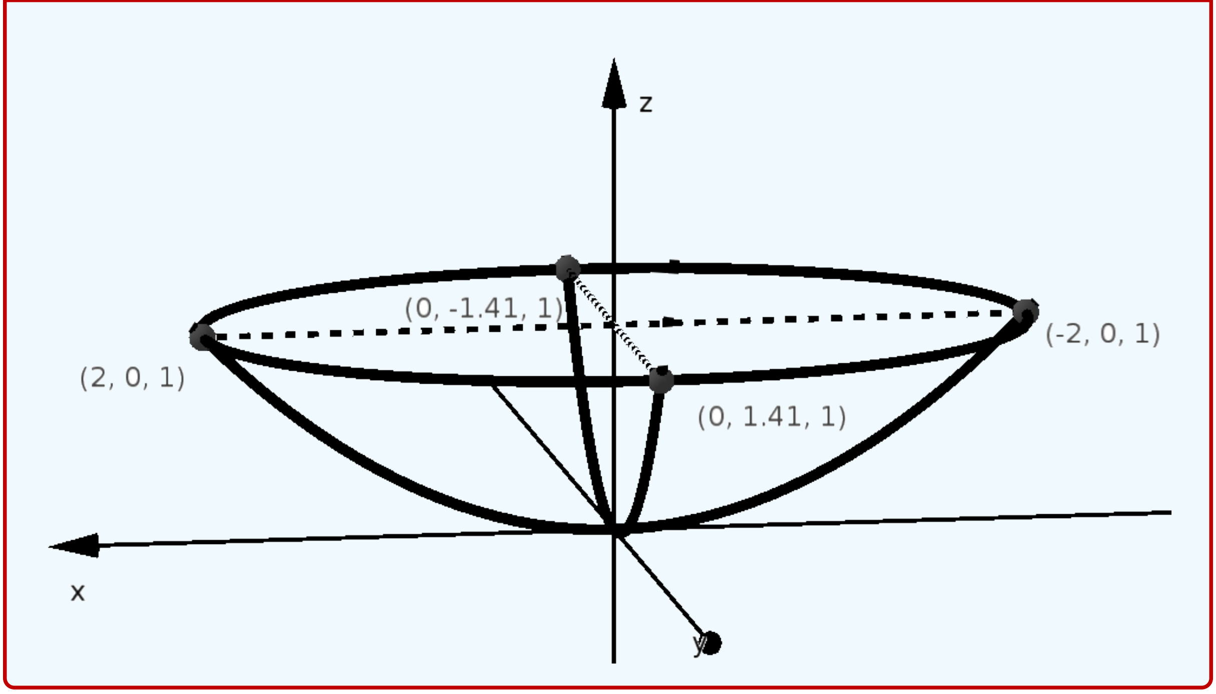
which is an ellipse with centre on the z-axis, major axis parallel to x-axis and $a = 2, b = \sqrt{2}$.

Traces in the xz (put $y = 0$) and yz (put $x = 0$) planes are

$$z = \frac{x^2}{4} \quad \text{and} \quad z = \frac{y^2}{2}.$$

They represent parabolas (as described in the case of (39)). Thus the graph is

Figure 5.13



■

A rough sketch of the hyperbolic paraboloid

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2} \quad (a > 0, b > 0)$$

can be obtained by first sketching the two parabolic traces that pass through the origin (one in the plane $x = 0$ and the other in the plane $y = 0$). After the parabolic traces are drawn, sketch the hyperbolic traces in the planes $z = \pm 1$ and then fill in any missing edges.

Problem 5.11

Sketch the graph of the hyperbolic paraboloid

$$z = \frac{y^2}{4} - \frac{x^2}{9}.$$

Solution. Setting $x = 0$ we get

$$z = \frac{y^2}{4} \quad (x = 0)$$

which is a parabola in the yz-plane with vertex at the origin and opening in the positive z-direction (since $z \geq 0$), and setting $y = 0$ yields

$$z = -\frac{x^2}{9} \quad (y = 0)$$

which is a parabola in the xz-plane with vertex at the origin and opening in the negative z-direction. The trace in the plane $z = 1$ is

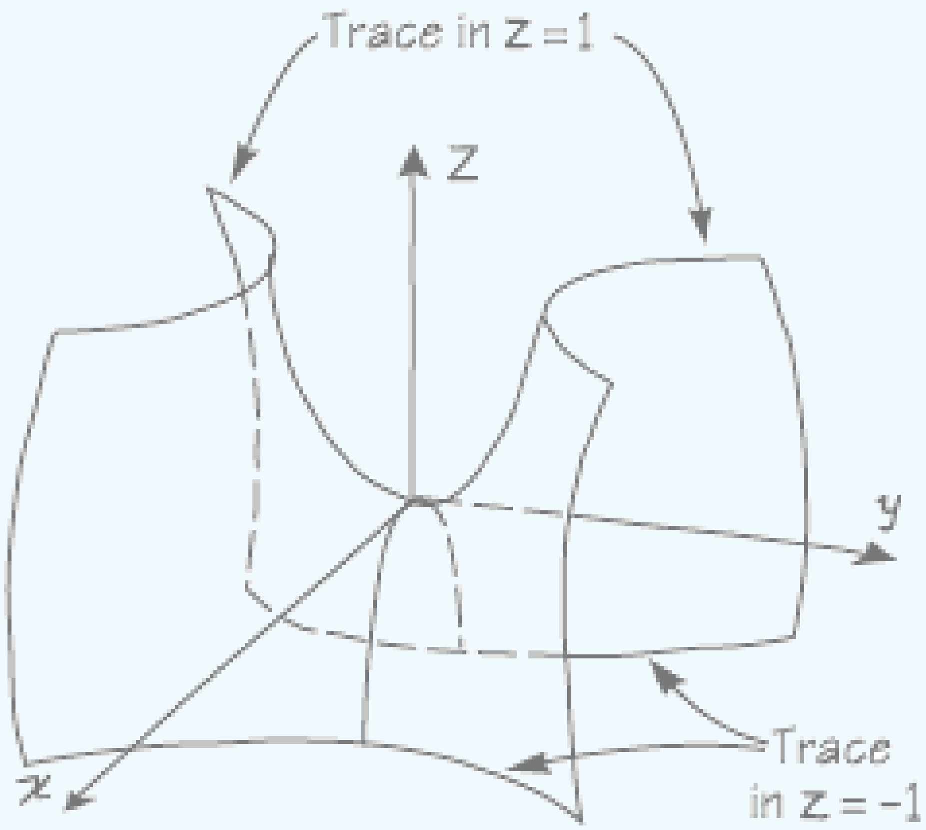
$$\frac{y^2}{4} - \frac{x^2}{9} = 1 \quad (z = 1)$$

which is a hyperbola that opens along a line parallel to the y-axis, and the trace in the plane $z = -1$ is

$$\frac{x^2}{9} - \frac{y^2}{4} = 1 \quad (z = -1)$$

which is a hyperbola that opens along a line parallel to the x-axis. Combining all of the above information leads to the sketch

Figure 5.14



■

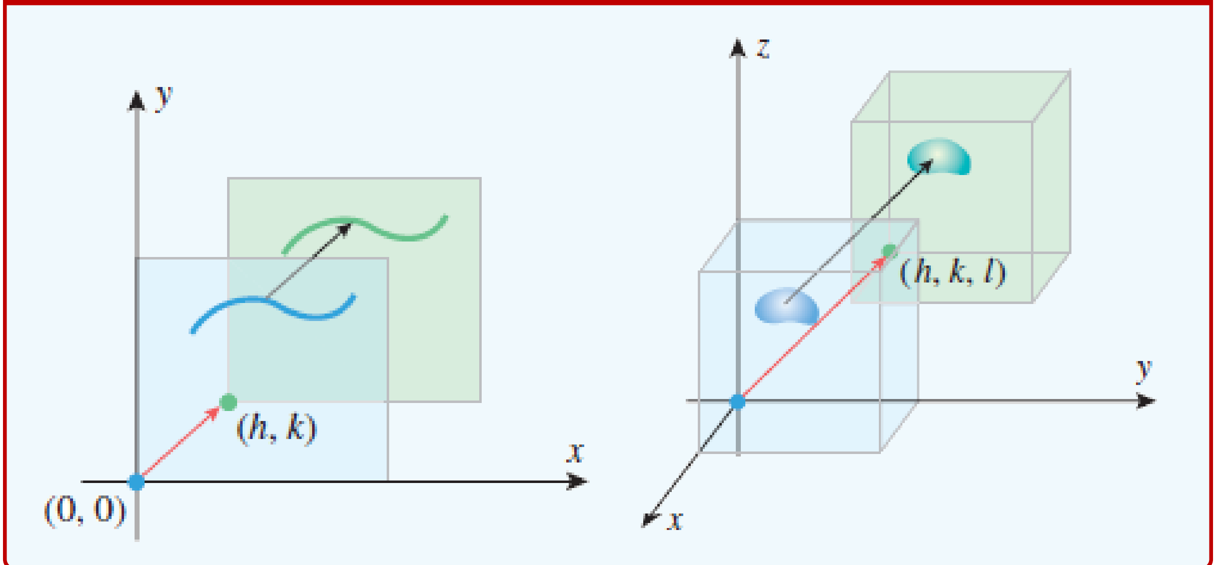
▶ The hyperbolic paraboloid in Figure 5.14 has an interesting behaviour at the origin -the trace in the xz-plane has a relative maximum at $(0, 0, 0)$, and the trace in the yz-plane has a relative minimum at $(0, 0, 0)$. Thus, a bug walking on the surface may view the origin as a highest point if travelling along one path, or may view the origin as a lowest point if travelling along a different path. A point with this property is commonly called a ***saddle point*** or a ***minimax point***.

TRANSLATIONS OF QUADRIC SURFACES

We saw that a conic in an xy-coordinate system can be translated by substituting $x - h$ for x and $y - k$ for y in its equation. To understand why this works, think of the xy-axes as fixed and think of the plane as a transparent sheet of plastic on which all graphs are drawn. When the coordinates of points are modified by substituting $(x-h, y-k)$ for (x, y), the geometric effect is to translate the sheet of plastic (and hence all curves) so that the point on the plastic that was initially at $(0,0)$ is moved to the point (h, k) (see first of Figure 5.15).

For the analog in three dimensions, think of the xyz-axes as fixed and think of 3-space as a transparent block of plastic in which all surfaces are embedded. When the coordinates of points are modified by substituting $(x-h, y-k, z-l)$ for (x, y, z), the geometric effect is to translate the block of plastic (and hence all surfaces) so that the point in the plastic block that was initially at $(0,0,0)$ is moved to the point (h, k, l).

Figure 5.15



Problem 5.12

Describe the surface

$$z = (x-1)^2 + (y+2)^2 + 3.$$

Solution. The equation can be rewritten as

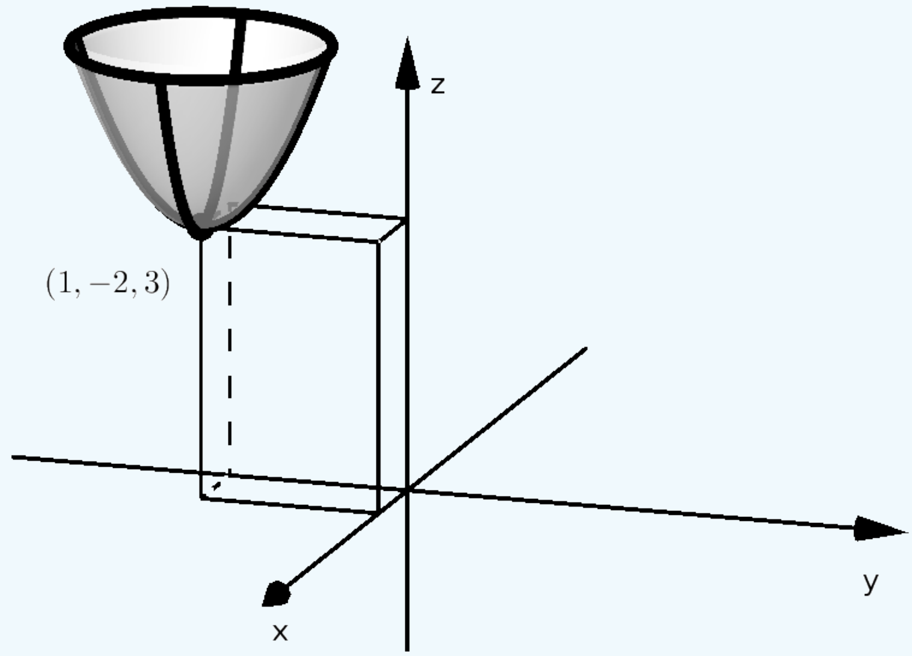
$$z - 3 = (x-1)^2 + (y+2)^2.$$

This surface is the paraboloid that results by translating the paraboloid

$$z = x^2 + y^2$$

so that the new "vertex" is at the point $(1, -2, 3)$. A rough sketch is

Figure 5.16



Problem 5.13

Describe the surface

$$4x^2 + 4y^2 + z^2 + 8y - 4z = -4.$$

Solution. Completing the squares we get

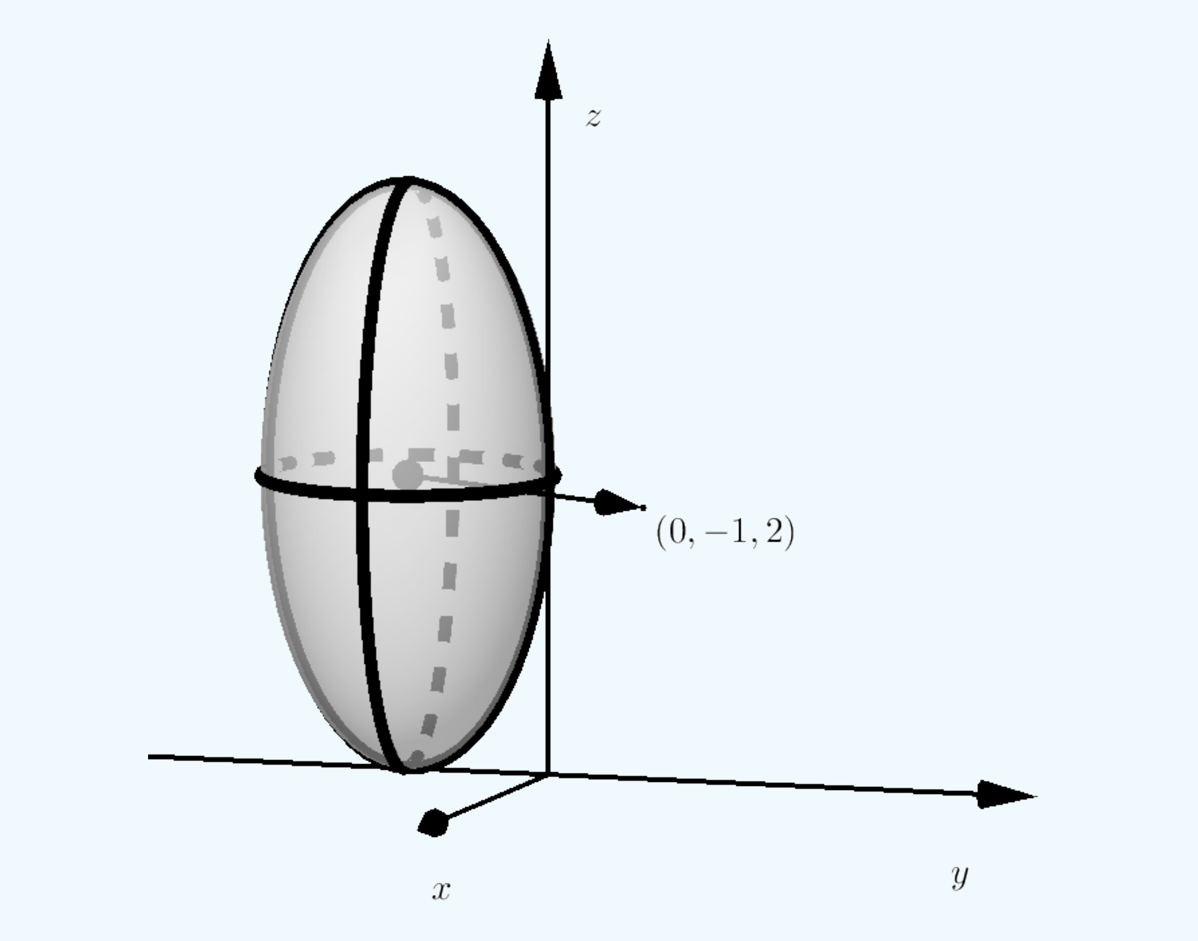
$$\begin{aligned} 4x^2 + 4(y^2 + 2y + 1) + (z^2 - 4z + 4) &= -4 + 4 + 4 \\ \Rightarrow 4x^2 + 4(y+1)^2 + (z-2)^2 &= 4 \\ \Rightarrow x^2 + (y+1)^2 + \frac{(z-2)^2}{4} &= 1. \end{aligned}$$

Thus, the surface is the ellipsoid that results when the ellipsoid

$$x^2 + y^2 + \frac{z^2}{4} = 1$$

is translated so that the new "center" is at the point $(0, -1, 2)$. A rough sketch of this ellipsoid is

Figure 5.17

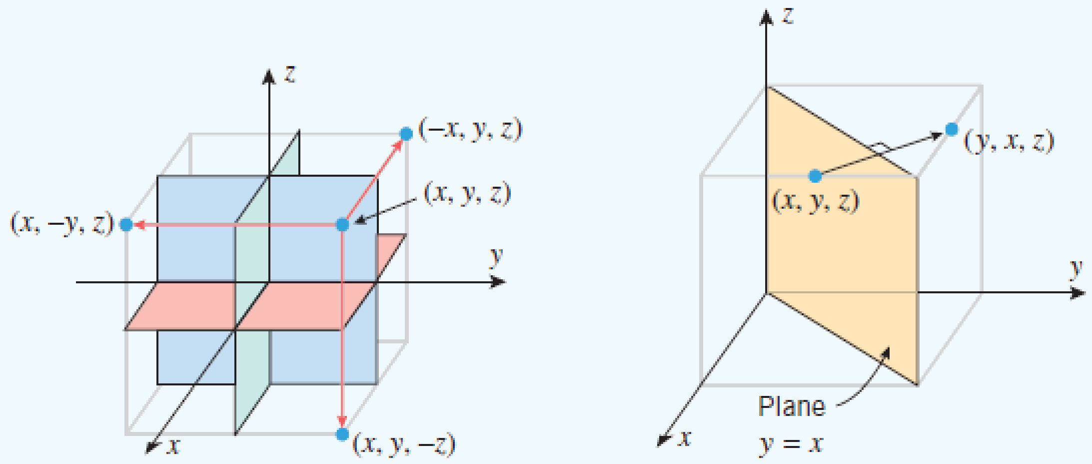


REFLECTIONS OF SURFACES IN 3-SPACE

Recall that in an xy-coordinate system a point (x, y) is reflected about the x-axis if y is replaced by $-y$, and it is reflected about the y-axis if x is replaced by $-x$. In an xyz coordinate system, a point (x, y, z) is reflected about the xy-plane if z is replaced by $-z$, it is reflected about the yz-plane if x is replaced by $-x$, and it is reflected about the xz-plane if y is replaced by $-y$ (first of Figure 5.18). It follows that replacing a variable by its negative in the equation of a surface causes that surface to be reflected about a coordinate plane.

Recall also that in an xy-coordinate system a point (x, y) is reflected about the line $y = x$ if x and y are interchanged. However, in an xyz-coordinate system, interchanging x and y reflects the point (x, y, z) about the plane $y = x$ (second of Figure 5.18). Similarly, interchanging x and z reflects the point about the plane $x = z$, and interchanging y and z reflects it about the plane $y = z$. Thus, it follows that *interchanging two variables in the equation of a surface reflects that surface about a plane that makes a* $45°$ *angle with two of the coordinate planes.*

Figure 5.18



Problem 5.14

Describe the surfaces

$$\text{(a) } y^2 = x^2 + z^2 \quad \text{(b) } z = -(x^2 + y^2)$$

Solution. (a) The graph of the equation

$$y^2 = x^2 + z^2$$

results from interchanging y and z in the equation

$$z^2 = x^2 + y^2.$$

Thus, the graph of the equation $y^2 = x^2 + z^2$ can be obtained by reflecting the graph of $z^2 = x^2 + y^2$ about the plane $y = z$. Since the graph of $z^2 = x^2 + y^2$ is a circular cone opening along the z-axis, it follows that the graph of $y^2 = x^2 + z^2$ is a circular cone opening along the y-axis.

(b) The graph of the equation

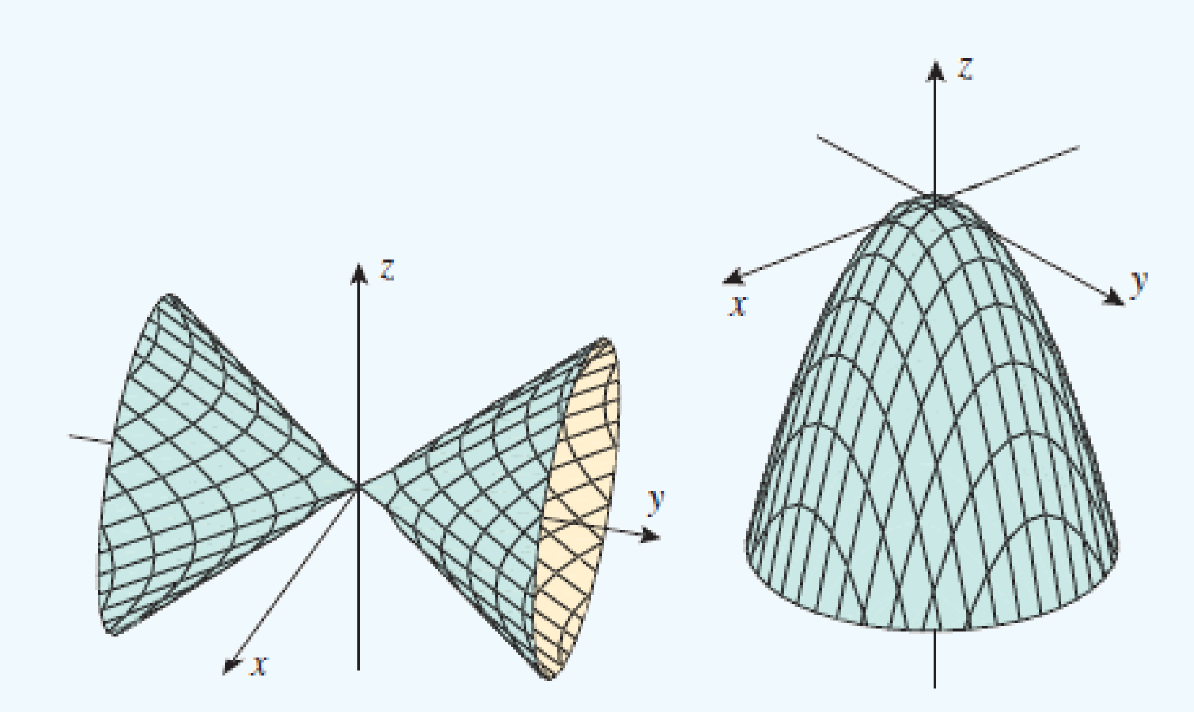
$$z = -(x^2 + y^2)$$

can be written as

$$-z = x^2 + y^2,$$

which can be obtained by replacing z with $-z$ in the equation $z = x^2 + y^2$. Since the graph of $z = x^2 + y^2$ is a circular paraboloid opening in the positive z-direction it follows that the graph of $z = -(x^2 + y^2)$ is a circular paraboloid opening in the negative z-direction.

Figure 5.19



The following table summarizes techniques for identifying quadric surfaces.

IDENTIFYING A QUADRIC SURFACE FROM THE FORM OF ITS EQUATION

EQUATION	CHARACTERISTIC	CLASSIFICATION
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	No minus signs	Ellipsoid
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	One minus sign	Hyperboloid of one sheet
$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	Two minus signs	Hyperboloid of two sheets
$z^2-\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	No linear terms	Elliptic cone
$z-\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	One linear term; two quadratic terms with the same sign	Elliptic paraboloid
$z-\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	One linear term; two quadratic terms with the opposite sign	Hyperbolic paraboloid

Problem 5.15

Identify the surfaces

$$\text{(a) } 3x^2 - 4y^2 + 12z^2 + 12 = 0 \quad \text{(b) } 4x^2 - 4y + z^2 = 0$$

Solution. (a) The equation can be rewritten as

$$\frac{y^2}{3} - \frac{x^2}{4} - z^2 = 1.$$

This equation has a 1 on the right side and two negative terms on the left side, so its graph is a hyperboloid of two sheets.
(b) The equation has one linear term and two quadratic terms with the same sign, so its graph is an elliptic paraboloid. ■

Problem 5.16

Identify the quadric surface as an ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, elliptic cone, elliptic paraboloid, or hyperbolic paraboloid. State the values of a, b, and c in each case.

1. (a) $z = \frac{x^2}{4} + \frac{y^2}{9}$
   (b) $z = \frac{y^2}{25} - x^2$
   (c) $x^2 + y^2 - z^2 = 16$
   (d) $x^2 + y^2 - z^2 = 0$
   (e) $4z = x^2 + 4y^2$
   (f) $z^2 - x^2 - y^2 = 1$
2. (a) $6x^2 + 3y^2 + 4z^2 = 12$
   (b) $y^2 - x^2 - z = 0$
   (c) $9x^2 + y^2 - 9z^2 = 9$
   (d) $4x^2 + y^2 - 4z^2 = -4$
   (e) $2z - x^2 - 4y^2 = 0$
   (f) $12z^2 - 3x^2 = 4y^2$

Solution. Comparing with the equations of quadrics and using techniques to identify them we shall identify the surfaces.

1. (a) Elliptic paraboloid (b) Hyperbolic paraboloid
   (c) Hyperboloid of one sheet (d) Elliptic cone (e) Elliptic paraboloid
   (f) Hyperboloid of two sheets

Other problems are left as exercise. ■

Problem 5.17

Find equations of the traces in the coordinate planes and sketch the traces in an xyz- coordinate system.

1. (a) $\frac{x^2}{9} + \frac{y^2}{25} + \frac{z^2}{4} = 1$

   (b) $z = x^2 + 4y^2$

   (c) $\frac{x^2}{9} + \frac{y^2}{16} - \frac{z^2}{4} = 1$

2. (a) $y^2 + 9z^2 = x$

   (b) $4x^2 - y^2 + 4z^2 = 4$

   (c) $z^2 = x^2 + \frac{y^2}{4}$

Solution. We obtain trace in the xy-plane by substituting $z = 0$, trace in the xz-plane by substituting $y = 0$, and trace in the yz-plane by substituting $x = 0$.

1. (a) When $z = 0$ we get, $\frac{x^2}{9} + \frac{y^2}{25} = 1$, which is an ellipse in xy-plane.
   When $y = 0$ we get, $\frac{x^2}{9} + \frac{z^2}{4} = 1$, which is an ellipse in xz-plane.
   When $x = 0$ we get, $\frac{y^2}{25} + \frac{z^2}{4} = 1$, which is an ellipse in yz-plane.

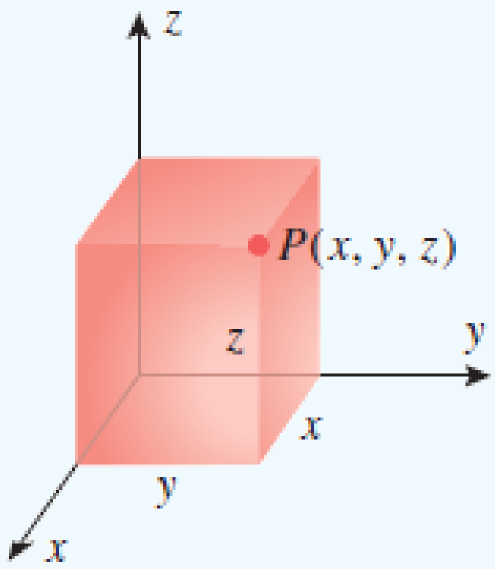
Other problems are left as exercise. ■

6 CYLINDRICAL AND SPHERICAL COORDINATES

We have discussed the rectangular coordinate system in 3-space. Now we discuss two new types of coordinate systems in 3-space that are often more useful than rectangular coordinate systems for studying surfaces with symmetries. These new coordinate systems also have important applications in navigation, astronomy, and the study of rotational motion about an axis.

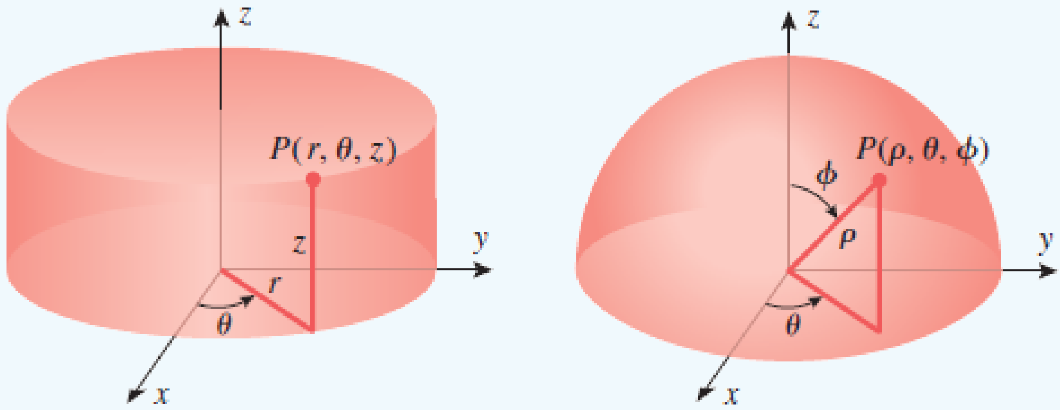
In rectangular coordinate system, a point $P(x, y, z)$ can be considered as a point on a vertex of a rectangular parallelepiped (or rectangular prism).

Figure 6.1



Consider a point P in the 3-space. Geometrically, P can be visualized as a point on the surface of a cylinder or a semi sphere. This leads to two new coordinate systems in 3-space namely, ***cylindrical coordinates*** and ***spherical coordinates***.

Figure 6.2



The cylindrical coordinates of P are

$$(r, \theta, z)$$

where z is the intercept of P from the z-axis, r, θ are the plane polar coordinates of the point at which the perpendicular from P intersect the xy-plane. In order to cover all points in the 3-space, we must have

$$-\infty < z < \infty, \quad r \geq 0, \quad 0 \leq \theta < 2\pi$$

The spherical coordinates of P are

$$(\rho, \theta, \phi)$$

where ρ is the distance of P from the origin, θ is the angular coordinate in plane polar coordinate of the point at which the perpendicular from P intersect the xy- plane and ϕ is the angle that the line joining origin and P makes with the positive z-axis. In order to cover all points in the 3-space, we must have

$$\rho \geq 0, \quad 0 \leq \theta < 2\pi, \quad 0 \leq \phi \leq \pi$$

CONSTANT SURFACES

In rectangular coordinates the surfaces represented by equations of the form

$$x = x_0, \quad y = y_0, \quad \text{and} \quad z = z_0$$

where x_0, y_0, and z_0 are constants, are planes parallel to the yz-plane, xz- plane, and xy-plane respectively (Figure 6.3(a)).

In cylindrical coordinates the surfaces represented by equations of the form

$$r = r_0, \quad \theta = \theta_0 \quad \text{, and} \quad z = z_0$$

where r_0, θ_0, and z_0 are constants, are shown in Figure 6.3(b).

- The surface $r = r_0$ is a right circular cylinder of radius r_0 centered on the z-axis.
- The surface $\theta = \theta_0$ is a half-plane attached along the z-axis and making an angle θ_0 with the positive x-axis.
- The surface $z = z_0$ is a horizontal plane.

In spherical coordinates the surfaces represented by equations of the form

$$\rho = \rho_0, \quad \theta = \theta_0, \quad \text{and} \quad \phi = \phi_0$$

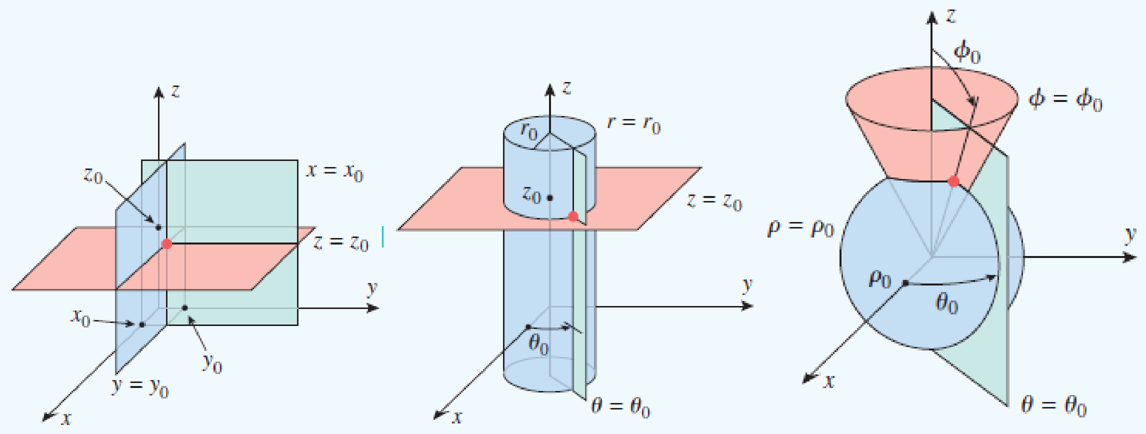
where ρ_0, θ_0, and ϕ_0 are constants, are shown in Figure 6.3(c).

- The surface $\rho = \rho_0$ consists of all points whose distance ρ from the origin is ρ_0. Assuming ρ_0 to be nonnegative, this is a sphere of radius ρ_0 centered at the origin.

- As in cylindrical coordinates, the surface $\theta = \theta_0$ is a half-plane attached along the z-axis, making an angle of θ_0 with the positive x-axis.

- The surface $\phi = \phi_0$ consists of all points from which a line segment to the origin makes an angle of ϕ_0 with the positive z-axis. If $0 < \phi_0 < \frac{\pi}{2}$, this will be the nappe of a cone opening up, while if $\frac{\pi}{2} < \phi_0 < \pi$, this will be the nappe of a cone opening down. (If $\phi_0 = \frac{\pi}{2}$, then the cone is flat, and the surface is the xy-plane.)

Figure 6.3



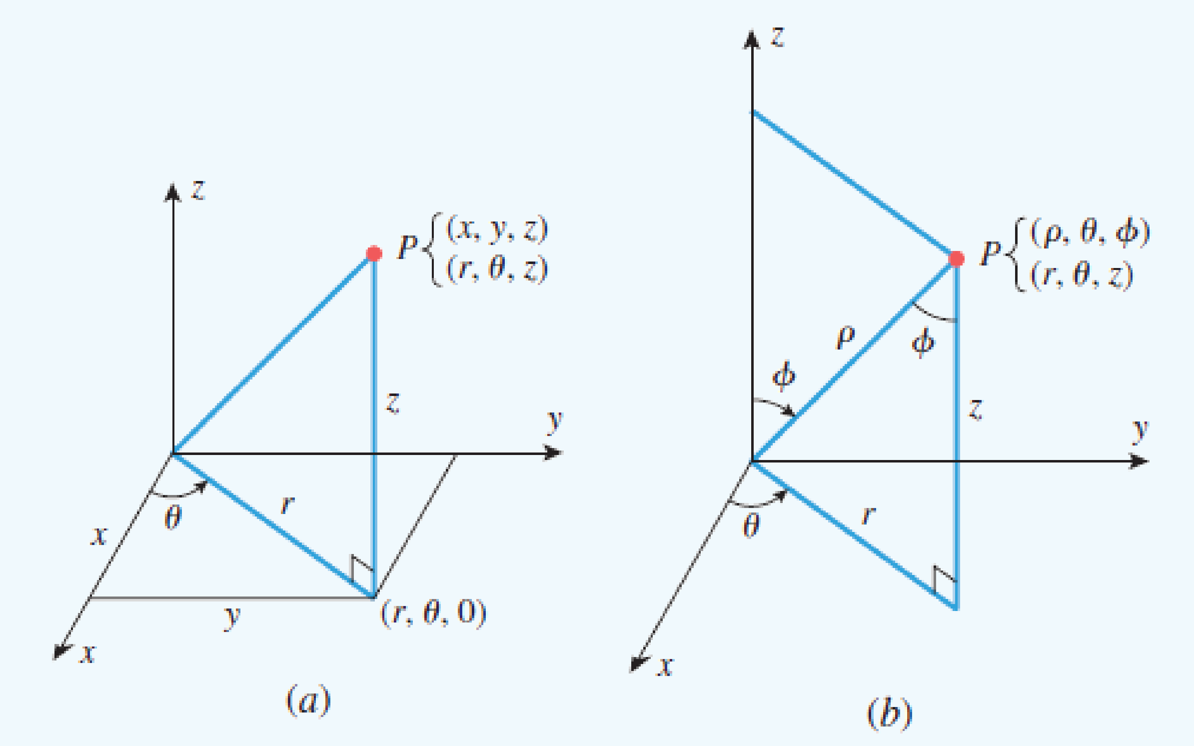
(a) (b) (c)

CONVERTING COORDINATES

Just as we needed to convert between rectangular and polar coordinates in 2-space, so we will need to be able to convert between rectangular, cylindrical, and spherical coordinates in 3-space. Following table provides formulas for making these conversions.

CONVERSION		FORMULAS
Cylindrical to rectangular	$(r, \theta, z) \to (x, y, z)$	$x = r\cos\theta, y = r\sin\theta, z = z$
Rectangular to cylindrical	$(x, y, z) \to (r, \theta, z)$	$r = \sqrt{x^2 + y^2}, \tan\theta = y/x, z = z$
Spherical to cylindrical	$(\rho, \theta, \phi) \to (r, \theta, z)$	$r = \rho\sin\phi, \theta = \theta, z = \rho\cos\phi$
Cylindrical to spherical	$(r, \theta, z) \to (\rho, \theta, \phi)$	$\rho = \sqrt{r^2 + z^2}, \theta = \theta, \tan\phi = r/z$
Spherical to rectangular	$(\rho, \theta, \phi) \to (x, y, z)$	$x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta,$ $z = \rho\cos\phi$
Rectangular to spherical	$(x, y, z) \to (\rho, \theta, \phi)$	$\rho = \sqrt{x^2 + y^2 + z^2}, \tan\theta = y/x,$ $\cos\phi = z/\sqrt{x^2 + y^2 + z^2}$

Figure 6.4



Cylindrical to Rectangular

From Figure 6.4(a), it is clear that $z = z$. Also since (r, θ) are the polar coordinates of (x, y) we get

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z \tag{43}$$

Rectangular to Cylindrical

From (44) we get

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}, \quad z = z. \tag{44}$$

Spherical to Cylindrical

In Figure 6.4(b), the right triangle with sides ρ, z, r shows that $r = \rho\sin\phi, z = \rho\cos\phi$. Hence we get

$$r = \rho\sin\phi, \quad \theta = \theta, \quad z = \rho\cos\phi. \tag{45}$$

Cylindrical to Spherical

From (45) we have $r^2 + z^2 = \rho^2(\sin^2\phi + \cos^2\phi) = \rho^2$ and $\tan\phi = \frac{r}{z}$. Hence

$$\rho = \sqrt{r^2 + z^2}, \quad \theta = \theta, \quad \tan\phi = \frac{r}{z}. \tag{46}$$

Spherical to Rectangular

In Figure 6.4(b), the right triangle with sides ρ, z, r shows that $r = \rho\sin\phi, z = \rho\cos\phi$. Also since $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi. \tag{47}$$

Rectangular to Spherical

From (47) we have, $x^2 + y^2 = \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta = \rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta) = \rho^2\sin^2\phi$ so that $x^2 + y^2 + z^2 = \rho^2\sin^2\phi + \rho^2\cos^2\phi = \rho^2(\sin^2\phi + \cos^2\phi) = \rho^2$. Also $y/x = (\rho\sin\phi\sin\theta)/(\rho\sin\phi\cos\theta) = \tan\theta$ and $\cos\phi = z/\rho$ so that, we have

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \tan\theta = \frac{y}{x}, \quad \cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \tag{48}$$

Problem 6.1

(a) Find the rectangular coordinates of the point with cylindrical coordinates

$$(r, \theta, z) = \left(4, \frac{\pi}{3}, -3\right)$$

(b) Find the rectangular coordinates of the point with spherical coordinates

$$(\rho, \theta, \phi) = \left(4, \frac{\pi}{3}, \frac{\pi}{4}\right).$$

Solution. (a) Use (44) for conversion. We have

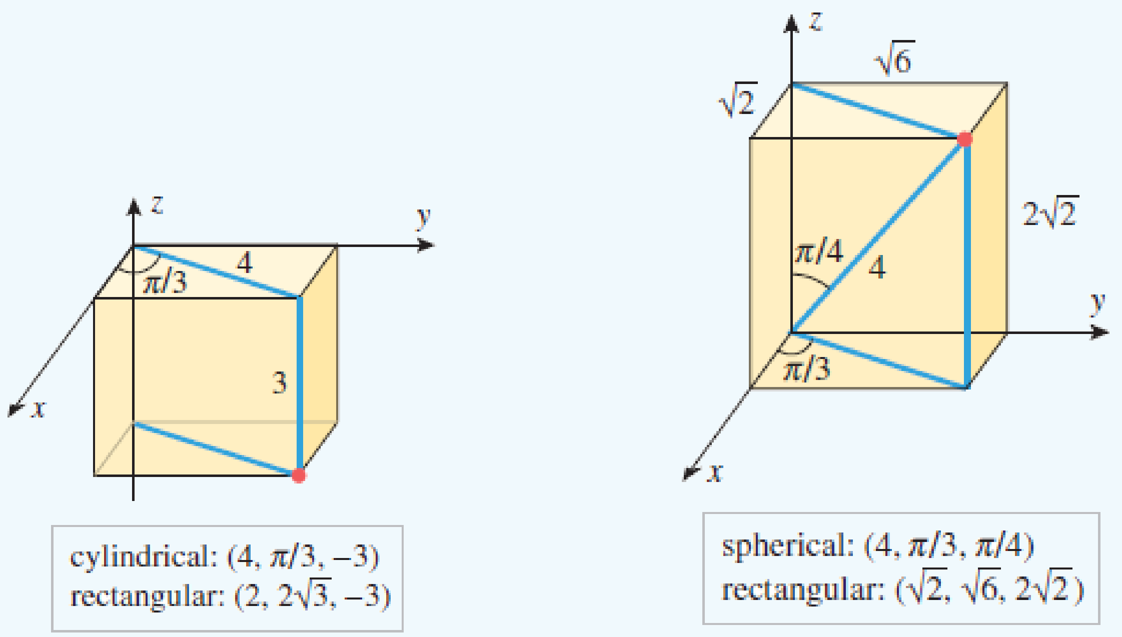
$$\begin{aligned}
x &= r\cos\theta = 4\cos\frac{\pi}{3} = 4 \times \frac{1}{2} = 2\\
y &= r\sin\theta = 4\sin\frac{\pi}{3} = 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}\\
z &= -3
\end{aligned}$$

Thus, the rectangular coordinates of the point are $(x, y, z) = (2, 2\sqrt{3}, -3)$.
(b) Use (47) for conversion. We have

$$\begin{aligned}
x &= \rho\sin\phi\cos\theta = 4\sin\frac{\pi}{4}\cos\frac{\pi}{3} = 4 \times \frac{1}{\sqrt{2}} \times \frac{1}{2} = \sqrt{2}\\
y &= \rho\sin\phi\sin\theta = 4\sin\frac{\pi}{4}\sin\frac{\pi}{3} = 4 \times \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} = \sqrt{6}\\
z &= \rho\cos\phi = 4\cos\frac{\pi}{4} = 4 \times \frac{1}{\sqrt{2}} = 2\sqrt{2}.
\end{aligned}$$

The rectangular coordinates of the point are $(x, y, z) = (\sqrt{2}, \sqrt{6}, 2\sqrt{2})$.

Figure 6.5



Problem 6.2

Find the spherical coordinates of the point that has rectangular coordinates

$$(x, y, z) = (4, -4, 4\sqrt{6})$$

Solution. Use (48) for conversion. We have

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{4^2 + (-4)^2 + (4\sqrt{6})^2} = \sqrt{128} = 8\sqrt{2}$$
$$\tan\theta = \frac{y}{x} = \frac{-4}{4} = -1$$
$$\cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{4\sqrt{6}}{8\sqrt{2}} = \frac{\sqrt{3}\sqrt{2}}{2\sqrt{2}} = \frac{\sqrt{3}}{2}$$

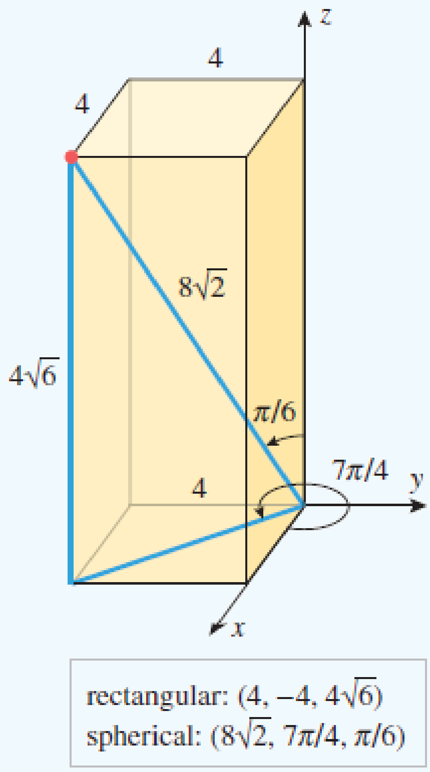
We have $\tan\theta = -1$. Since $0 \le \theta < 2\pi$ and $x > 0, y < 0$, the point $(x, y, 0)$ lies on the fourth quadrant of xy-plane. Hence we get

$$\theta = 2\pi - \tan^{-1}(1)2\pi - \frac{\pi}{4} = \frac{7\pi}{4}.$$

Also $\cos\phi = \frac{\sqrt{3}}{2}$ and $0 \le \phi \le \pi$ so that $\phi = \frac{\pi}{6}$. Thus, the spherical coordinates of the point are

$$(\rho, \theta, \phi) = \left(8\sqrt{2}, \frac{7\pi}{4}, \frac{\pi}{6}\right)$$

Figure 6.6



EQUATIONS OF SURFACES IN CYLINDRICAL AND SPHERICAL COORDINATES

Surfaces of revolution about the z-axis of a rectangular coordinate system usually have simpler equations in cylindrical coordinates than in rectangular coordinates, and the equations of surfaces with symmetry about the origin are usually simpler in spherical coordinates than in rectangular coordinates. For example, consider the upper nappe of the circular cone whose equation in rectangular coordinates is

$$z = \sqrt{x^2 + y^2}$$

The corresponding equation in cylindrical coordinates can be obtained from the cylindrical-to-rectangular conversion formulas in (44). This yields

$$z = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} = \sqrt{r^2} = |r| = r$$

so the equation of the cone in cylindrical coordinates is

$$z = r.$$

Similarly, the equation of the cone in spherical coordinates can be obtained as

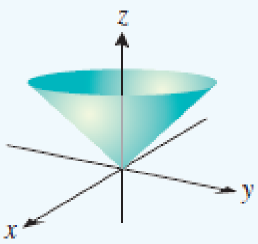
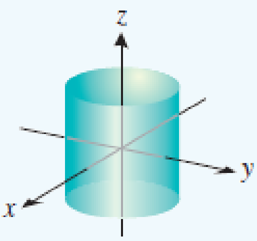
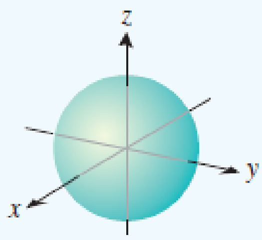
$$\rho\cos\phi = \rho\sin\phi$$

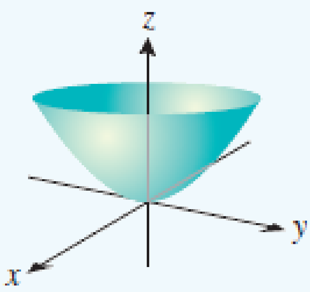
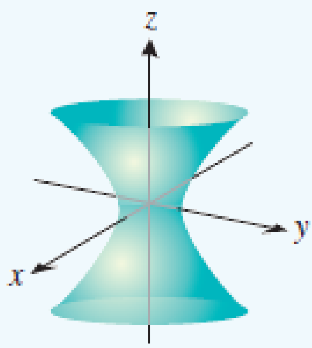
which, if $\rho \neq 0$, can be rewritten as

$$\tan\phi = 1 \quad \text{or} \quad \phi = \frac{\pi}{4}$$

Geometrically, this tells us that the radial line from the origin to any point on the cone makes an angle of $\frac{\pi}{4}$ with the z-axis.

Figure 6.7

	CONE	CYLINDER	SPHERE
			
RECTANGULAR	$z = \sqrt{x^2 + y^2}$	$x^2 + y^2 = 1$	$x^2 + y^2 + z^2 = 1$
CYLINDRICAL	$z = r$	$r = 1$	$z^2 = 1 - r^2$
SPHERICAL	$\phi = \pi/4$	$\rho = \csc\phi$	$\rho = 1$

	PARABOLOID	HYPERBOLOID
		
RECTANGULAR	$z = x^2 + y^2$	$x^2 + y^2 - z^2 = 1$
CYLINDRICAL	$z = r^2$	$z^2 = r^2 - 1$
SPHERICAL	$\rho = \cos\phi\csc^2\phi$	$\rho^2 = -\sec 2\phi$

Problem 6.3

Find equations of the paraboloid

$$z = x^2 + y^2$$

in cylindrical and spherical coordinates.

Solution. Converting to cylindrical coordinates we get

$$z = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2.$$

Converting to cylindrical coordinates we get

$$\begin{aligned}\rho\cos\phi &= \rho^2\sin^2\phi\\ \Rightarrow \rho &= \frac{\cos\phi}{\sin^2\phi}\\ &= \cos\phi\,\mathrm{cosec}^2\phi.\end{aligned}$$

■

REFERENCES

[1] Howard Anton, Irl Bivens, Stephen Davis, *Calculus*, 10^{th} Edition, JOHN WILEY & SONS, INC.

[2] G B Thomas, R L Finney, *Calculus*, 9th Edition, Addison-Weseley Publishing Company.

[3] J Stewart, *Calculus with Early Transcendental Functions*, 7th Edition, Cengage India Private Limited.